NONLINEAR ELLIPTIC PROBLEMS WITH DYNAMICAL BOUNDARY CONDITIONS OF REACTIVE AND REACTIVE-DIFFUSIVE TYPE

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Abstract. We investigate classical solutions of nonlinear elliptic equations with two classes of dynamical boundary conditions, of reactive and reactive-diffusive type. In the latter case it is shown that well-posedness is to a large extent independent of the coupling with the elliptic equation. For both types of boundary conditions we consider blow-up, global existence, global attractors and convergence to single equilibria.

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1. Introduction

The prototype of the elliptic-parabolic initial-boundary value problems that we consider in this article is

$$\begin{cases} \lambda u - d\Delta u = f(u) & \text{in } (0,T)\times\Omega, \\ \partial_t u_\Gamma - \delta\Delta_\Gamma u_\Gamma + d\partial_\nu u = g(u_\Gamma) & \text{on } (0,T)\times\Gamma, \\ u|_\Gamma = u_\Gamma & \text{on } (0,T)\times\Gamma, \\ u|_{t=0} = u_0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

We assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$, that $d > 0$, $\delta \geq 0$ and $f, g \in C^\infty(\mathbb{R})$. Further, Δ_Γ is the Laplace-Beltrami operator and ∂_ν is the outer normal derivative on Γ. It is throughout assumed that f is globally Lipschitz continuous and that λ is sufficiently large, in dependence on f. Depending on δ, two classes of boundary conditions are modelled by (1.1). For $\delta > 0$ we have boundary conditions of reactive-diffusive type, and for $\delta = 0$ the boundary conditions are purely reactive.

The motivation to consider (1.1) comes from physics. The function u represents the steady state temperature in a body Ω such that the rate at which u evolves through the boundary Γ is proportional to the flux on the boundary, up to some correction $\delta\Delta_\Gamma u_\Gamma$, $\delta \geq 0$, which from a modelling viewpoint, accounts for small diffusive effects along Γ. Moreover, the heat source on Γ acts nonlinearly through the function g. Problem (1.1) is also important in conductivity (see, e.g., [17]) and harmonic analysis due to its connection to the following eigenvalue problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad -\delta\Delta_\Gamma u_\Gamma + \partial_\nu u = \xi u_\Gamma \quad \text{on } \Gamma, \tag{1.2}$$

which was introduced by Stekloff [33] (initially) in the case $\delta = 0$. This connection arises because the linear problem associated with (1.1) (i.e., by letting $\lambda = 0$, $f \equiv 0$ and $g \equiv 0$) can be solved by the Fourier method in terms of the eigenfunctions of (1.2) (see [37], which also includes the case $\delta > 0$; cf. also [38] for $\delta = 0$). The solvability of the linear problem (assuming $\delta = 0$) was also investigated by Hintermann [19] by means of the theory of pseudo-differential operators, and by Gröger [17] and Showalter [32], by applying the theory of maximal monotone operators in the Hilbert-space setting (see, also, [12]). It turns out that this connection is also essential for solvability of the nonlinear problem (1.1).

The mathematical study of the prototype (1.1) has a long-standing history. In [22] J.-L. Lions considered the special case $\delta = \lambda = 0$, $f \equiv 0$ and $g(s) = -|s|^p s$, $p > 0$. By standard compactness methods, he proved existence and uniqueness of global solutions for initial datum $u_0 \in H^{1/2}(\Gamma)$ in this special case. Problem (1.1) was investigated in the general case by Escher [7, 8] for nontrivial functions f, g, by also treating systems of elliptic equations, but always in the case $\delta = 0$. His papers deal with classical solvability and global existence for smooth initial data. In particular, global existence of classical solutions was shown assuming f is globally Lipschitz and that $g(s)\, s \leq 0$, for all $s \in \mathbb{R}$. Constantin, Escher and Yin [4, 43] established, in the case $\delta = \lambda = 0$ and $f \equiv 0$, some natural structural conditions for the function g so that global existence of classical solutions holds. Their approach is based on global existence criteria for ODEs. Boundedness of the global solutions for (1.1) was shown by Fila and Quittner [11] in the case when $\delta = \lambda = 0$, $f \equiv 0$ and g is a superlinear *subcritical* nonlinearity. They have also proved that blow-up in finite time occurs for (1.1) if $g(s) = |s|^{p-1} s - as$, $p > 1$, $a \geq 0$ and if the initial datum u_0 is "large" enough [11, Section 3]. Blow-up phenomena for smooth

solutions of (1.1), when $\delta = 0$ and $f \equiv 0$, was also observed by Kirane [20] under some general assumptions on g, i.e., when $g(s) > 0$, for all $s \geq s_0$, and

$$\int_{s_0}^{\infty} \frac{d\xi}{g(\xi)} < \infty.$$

A version of the problem (1.1) for which the dynamic boundary condition is replaced by

$$|\partial_t u_\Gamma|^{m-1} \partial_t u_\Gamma + d\partial_\nu u = |u_\Gamma|^{p-1} u_\Gamma \qquad \text{on } \Gamma \times (0,T),$$

for some $m \geq 1$ and $p \geq 1$ was investigated by Vitillaro [38] for initial data $u_0 \in H^{1/2}(\Gamma)$ and $f \equiv 0$. He mainly devotes his attention to proving the local and the global existence as well as blow-up of solutions for $m \geq 1$, especially, in the nonlinear case when $m \neq 1$. Finally, it is interesting to note that, in the case when $f \neq 0$ but f is *not* globally Lipschitz, global non-existence without blow-up and non-uniqueness phenomena for (1.1) can occur (see [10]).

All the papers quoted so far deal only with classical issues, such as global existence, uniqueness and blow-up phenomena for (1.1) when $\delta = 0$. Concerning further regularity and longtime behavior of solutions, as time goes to infinity, not much seems to be known. This seems to be due to the fact that the gradient structure of (1.1) has not been exploited before. This issue is intimately connected with a *key* result on smoothness in $\mathbb{R}_+ \times \overline{\Omega}$ of solutions for (1.1) even when $f \neq 0$ (see Proposition 5.2), which is essential to the study of the asymptotic behavior of the system, in terms of global attractors and ω-limit sets.

The main novelties with respect to previous results on (1.1) are the following:

(**i**) The local well-posedness results are extended to the case $\delta > 0$. In fact, we will consider a more general class of elliptic problems with quasilinear, nondegenerate dynamic boundary conditions of reactive-diffusive type. More precisely, we consider the following generalization of the prototype model (1.1),

$$\begin{cases} \lambda u + \mathcal{A}u = f(u) & \text{in } (0,T) \times \Omega, \\ \partial_t u_\Gamma + \mathcal{C}(u_\Gamma)u_\Gamma + \mathcal{B}(u) = g(u_\Gamma) & \text{on } (0,T) \times \Gamma, \\ u|_{t=0} = u_0 & \text{on } \Gamma, \end{cases} \tag{1.3}$$

where

$$\mathcal{A}u = -\mathrm{div}\big(d\nabla u\big), \qquad \mathcal{C}(u_\Gamma)u_\Gamma = -\mathrm{div}_\Gamma\big(\delta(\cdot, u_\Gamma)\nabla_\Gamma u_\Gamma\big),$$

such that $d \in C^\infty(\overline{\Omega})$, $\delta \in C^\infty(\Gamma \times \mathbb{R})$ with $d \geq d_* > 0$ and $\delta \geq \delta_* > 0$. Moreover, ∇_Γ is the surface gradient and div_Γ is the surface divergence. Here and in the sequel we always assume that $u|_\Gamma = u_\Gamma$. The nonlinear map $\mathcal{B}$ in (1.3) couples the equations in the domain Ω and on the boundary Γ in a (possibly) nontrivial way. We do *not* impose any further structural conditions for $\mathcal{B}$ and g other than they must be of order strictly lower than two and satisfy a local Lipschitz condition. One example for $\mathcal{B}$ we have in mind is $\mathcal{B}(u) = b\nu \cdot (\nabla u)|_\Gamma$, with *no* sign restriction on $b \in C^\infty(\Gamma)$. We prove that for sufficiently large λ and a *globally* Lipschitz function f the problem (1.3) generates a (compact) local semiflow of solutions for $u_0 \in \mathcal{X}_\delta := W^{2-2/p,p}(\Gamma)$, $p \in (n+1, \infty)$, $\delta > 0$, and establish some further regularity properties for the local solution $u = u(\cdot; u_0)$. For the notion of local semiflow, we refer the reader to Section 2.2.

The independence of the well-posedness of the coupling was first observed by Vazquez and Vitillaro [37] for a linear model problem with $\mathcal{C} = -\Delta_\Gamma$ and $\mathcal{B} = -\partial_\nu$ in a Hilbert space setting. Our approach to the quasilinear problem is based on

maximal L^p-regularity properties of the corresponding linearized dynamic equation on the boundary. In Section 3 these will be verified for a general class of elliptic boundary differential operators using localization techniques. The global Lipschitz condition on f allows to solve the elliptic equation on Ω and to rewrite (1.3) as an initial-value problem for u_Γ on Γ, which can be treated with the general theory of [21]. The fact that the concrete form of the coupling $\mathcal{B}$ is inessential is a consequence of the fact that maximal regularity is invariant under lower order perturbations. For the precise statements of these results we refer the reader to Section 4.

The corresponding result for boundary conditions of purely reactive type, i.e., $\mathcal{C} \equiv 0$ and $\mathcal{B} = d\partial_\nu$ in (1.3), was shown in [7]. There the result is based on the generation properties of the Dirichlet-Neumann operator and thus, the solutions enjoy worse regularity properties up to $t = 0$. In addition to this we establish the compactness of the solution semiflow on $\mathcal{X}_0 := W^{1-1/p,p}(\Gamma)$, $p \in (n, \infty)$, in this case (see Section 4.3).

(**ii**) The blow-up results for problem (1.1), from [20] and [39], are also extended to the case when $\delta > 0$ and $f \neq 0$. Our approach is based on the method of subsolutions and a comparison lemma, and is inspired by [2] and [31] (see Section 5.2). We further show global existence of solutions of (1.1) under the natural assumption that $g(\xi)\xi \leq c_g(|\xi|^2 + 1)$ for all $\xi \in \mathbb{R}$ by performing a Moser-Alikakos iteration procedure as in [13, 26]. Here an inequality of Poincaré-Young type allows to connect the structure of the elliptic equation with that of the dynamic equation on Γ (see Section 5.3).

(**iii**) We prove the smoothness of solutions of (1.1) in both space and time exploiting a variation of parameters formula for the trace u_Γ, which is entirely new (see Section 5.1). Consequently, taking advantage of this smoothness, we can show that (1.1) has a gradient structure, and as a result establish the existence of a finite-dimensional global attractor in the phase space $\mathcal{X}_\delta$ for both types of boundary conditions. Here the main assumption is that the first eigenvalue of a Stekloff-like eigenvalue problem (similar to (1.2)) is positive (see Section 5.4).

(**iv**) The ω-limit sets of (1.1) can exhibit a complicated structure if the functions f, g are non-monotone and, a fortiori, the same is true for the global attractor. Indeed, when f, g are non-monotone (i.e., the related potentials $F(s) = \int_0^s f(y)\,dy$, $G(s) = \int_0^s g(y)\,dy$ are non-convex) this can happen if the stationary problem associated with (1.1) possesses a continuum of nonconstant solutions. Some examples which show that the ω-limit set can be a continuum are provided in [28]. However, assuming the nonlinearities f, g to be real analytic, we prove the convergence of a given trajectory $u = u(t; u_0)$, $u_0 \in \mathcal{X}_\delta$, as time goes to infinity, to a single equilibrium of (1.1). This shows, in a strong form, the asymptotic stability of $u(t; u_0)$ for an arbitrary (but given) initial datum $u_0 \in \mathcal{X}_\delta$. This type of result exploits a technique which is based on the so-called Łojasiewicz-Simon inequality (see Section 5.5; cf. also [34, 42]).

Finally, it is worth mentioning that most of our results can be also extended to systems of nonlinear elliptic equations subject to both types of boundary conditions.

The *plan of the paper* goes as follows. In Section 2, we introduce the functional analytic framework associated with (1.1) and (1.3), respectively. In Section 3, maximal L^p-regularity theory is developed for elliptic boundary differential operators of second order. Then, in Section 4 (and corresponding subsections) we prove (local) well-posedness results for (1.3) and establish the existence of compact (local)

semiflow on the corresponding phase spaces for each $\delta \geq 0$. The final Section 5 is further divided into five parts: the first part provides the key result which shows the smoothness of solutions in both space and time, while the second and third parts deal with blow-up phenomena and global existence, respectively. Finally, the last two subsections deal with the asymptotic behavior as time goes to infinity, in terms of global attractors and convergence of solutions to single equilbria.

2. Preliminaries

2.1. **Function spaces.** We briefly describe the function spaces that are used in the paper. Details and proofs can be found in [24, 36].

Throughout, all function spaces under consideration are real. Let $p \in [1, \infty]$. If $\Omega \subseteq \mathbb{R}^n$ is open, we denote by $L^p(\Omega)$ the usual Lebesgue spaces. Now let Ω have a (sufficiently) smooth boundary. Then for $s \geq 0$ and $p \in [1, \infty)$ we denote by $H^{s,p}(\Omega)$ the Bessel-potential spaces and by $W^{s,p}(\Omega)$ the Slobodetskij spaces. One has $H^{s,2}(\Omega) = W^{s,2}(\Omega)$ for all s, but for $p \neq 2$ the identity $H^{s,p}(\Omega) = W^{s,p}(\Omega)$ is only true if $s \in \mathbb{N}_0$. If $s \in \mathbb{N}_0$, then $H^{s,p}(\Omega)$ and $W^{s,p}(\Omega)$ coincide with the usual Sobolev spaces. In the case of noninteger differentiability, for our purposes it suffices to consider these spaces as interpolation spaces. If $s = [s] + s_* \notin \mathbb{N}_0$ with $[s] \in \mathbb{N}_0$ and $s_* \in (0, 1)$, then

$$H^{s,p} = [H^{[s],p}, H^{[s]+1,p}]_{s_*}, \qquad W^{s,p} = (W^{[s],p}, W^{[s]+1,p})_{s_*,p}, \tag{2.1}$$

where $[\cdot,\cdot]_{s_*}$ and $(\cdot,\cdot)_{s_*,p}$ denote complex and real interpolation, respectively. Moreover, $H^{s,p} = [L^p, H^{2,p}]_{s/2}$ for $s \in (0, 2)$ and $W^{s,p}(\Omega) = (L^p, W^{2,p})_{s,p}$ for $s \in (0, 2)$, $s \neq 1$. A useful tool are interpolation inequalities. We shall make particular use of

$$\|u\|_{H^{s,p}} \leq \|u\|_{L^p}^{1-s/2} \|u\|_{H^{2,p}}^{s/2}, \qquad \|u\|_{W^{s,p}} \leq C\, \|u\|_{L^p}^{1-s/2} \|u\|_{W^{2,p}}^{s/2}, \tag{2.2}$$

which is valid for all $u \in H^{2,p} = W^{2,p}$.

The corresponding function spaces over the boundary $\Gamma = \partial\Omega$ of a bounded smooth domain $\Omega \subset \mathbb{R}^n$ are defined via local charts. Let $\mathrm{g}_i : U_i \subset \mathbb{R}^{n-1} \to \Gamma$ be a finite family of parametrizations such that $\bigcup_i \mathrm{g}_i(U_i)$ covers Γ, and let $\{\psi_i\}$ be a partition of unity for Γ subordinate to this cover. Then for $s \geq 0$ we have

$$H^{s,p}(\Gamma) = \big\{ u \in L^p(\Gamma) \;:\; (\psi_i u) \circ \mathrm{g}_i \in H^{s,p}(\mathbb{R}^{n-1}) \text{ for all } i \big\},$$

and an equivalent norm is given by $\|u\|_{H^{s,p}(\Gamma)} = \sum_i \|(\psi_i u) \circ \mathrm{g}_i\|_{H^{s,p}(\mathbb{R}^{n-1})}$. The spaces $W^{s,p}(\Gamma)$ are defined in the same way, replacing H by W. In this way the properties of the spaces over Ω described above carry over to the spaces over Γ.

For $p \in (1, \infty)$ and $s > 1/p$ the trace $\operatorname{tr} u = u|_\Gamma$ extends to a continuous operator

$$\operatorname{tr} : H^{s,p}(\Omega) \to W^{s-1/p,p}(\Gamma).$$

Here we exclude the case $s - 1/p \in \mathbb{N}$ for $p \neq 2$.

2.2. **Semiflows.** Let $\mathcal{X}$ be a Banach space and let $t^+ : \mathcal{X} \to (0, \infty]$ be lower semicontinuous. Then we call a map

$$S : \bigcup_{x \in \mathcal{X}} [0, t^+(x)) \times \{x\} \to \mathcal{X}$$

a local semiflow on $\mathcal{X}$ if for all $x \in \mathcal{X}$ it holds that $S(\cdot; x) : [0, t^+(x)) \to \mathcal{X}$ is continuous, if $S(t, \cdot) : B_r(x) \subset \mathcal{X} \to \mathcal{X}$ is continuous for $t < t^+(x)$ and sufficiently small $r > 0$, if $S(0; \cdot) = \mathrm{id}_\mathcal{X}$, $S(t + s; x) = S(t; S(s; x))$ and if $t^+(x) < \infty$ implies that $\|S(t; x)\|_\mathcal{X} \to \infty$ as $t \to t^+$. In addition we call S compact, if for all bounded

sets $M \subset \mathcal{X}$ with $t^+(M) \geq T > 0$ and all $t \in (0,T)$ it holds that $S(t;M)$ is relatively compact in $\mathcal{X}$.

If $t^+(x) = \infty$ for all $x \in \mathcal{X}$, then we call S a global semiflow. In this case our notion of a semiflow coincides with the one in [3].

Note that, in contrast to parts of the literature, we include the condition for global existence (i.e., $t^+ = \infty$) already in the definition of a local semiflow.

3. Maximal L^p-regularity for boundary differential operators

In this section we show maximal L^p-regularity for elliptic boundary differential operators of second order.

3.1. **Boundary differential operators.**

Throughout, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. We describe our notion of a differential operator on Γ with possibly nonsmooth coefficients.

Let $(0,T)$ be a finite or infinite time interval. We call a globally defined, linear map $\mathcal{C} : (0,T) \times C^\infty(\Gamma) \to L^1(\Gamma)$ a (non-autonomous) *boundary differential operator* of order $k \in \mathbb{N}$, if for all $t \in (0,T)$ and all parametrizations $\mathrm{g} : U \subset \mathbb{R}^{n-1} \to \Gamma$ it holds

$$(\mathcal{C}(t,\cdot)u) \circ \mathrm{g}(x) = \sum_{|\gamma| \leq k} c_\gamma^{\mathrm{g}}(t,x) D_{n-1}^\gamma (u \circ \mathrm{g})(x), \qquad x \in U,$$

with local coefficients $c_\gamma^{\mathrm{g}}(t,\cdot) \in L^1(U)$ and $D_{n-1} = -\mathrm{i}\nabla_{n-1}$. The coefficients do not have to be globally defined and may in fact depend on the parametrization g. The examples we have in mind are the Laplace-Beltrami operator $\Delta_\Gamma = \mathrm{div}_\Gamma \nabla_\Gamma$, which is in coordinates given by

$$(\Delta_\Gamma u) \circ \mathrm{g} = \frac{1}{\sqrt{|\mathrm{G}|}} \sum_{i,j=1}^{n-1} \partial_i \big(\sqrt{|\mathrm{G}|}\mathrm{g}^{ij} \partial_j (u \circ \mathrm{g})\big),$$

and, for a tangential vector field $\mathcal{V}$ on Γ, a surface convection term $\mathcal{V}\nabla_\Gamma$, i.e.,

$$(\mathcal{V}\nabla_\Gamma u) \circ \mathrm{g} = \sum_{i,j=1}^{n-1} \mathrm{g}^{ij} (\mathcal{V} \cdot \partial_i \mathrm{g}) \partial_j (u \circ \mathrm{g}).$$

Here $\mathrm{G}^{-1} = (\mathrm{g}^{ij})_{i,j}$ is the inverse of the fundamental form G corresponding to g.

As in the euclidian case, the regularity of the local coefficients c_γ^{g} decides on which scale of function spaces over Γ the operator $\mathcal{C}(t,\cdot)$ acts. For instance, if $c_\gamma^{\mathrm{g}}(t,\cdot) \in L^\infty(U)$ for all parametrizations g and all γ, then we obtain for all $p \in [1,\infty]$ an estimate

$$\|\mathcal{C}(t,\cdot)u\|_{L^p(\Gamma)} \leq C\, \|u\|_{W^{k,p}(\Gamma)}, \qquad u \in C^\infty(\Gamma).$$

In this case $\mathcal{C}(t,\cdot)$ extends uniquely to a bounded linear map $W^{k,p}(\Gamma) \to L^p(\Gamma)$, or to a closed operator on $L^p(\Gamma)$ with domain $W^{k,p}(\Gamma)$. Of course, in view of Sobolev embeddings, for such an extension the regularity of the coefficients can be lowered in many cases.

Finally, structural conditions like ellipticity of a boundary differential operator $\mathcal{C}$ can also be imposed to hold locally with respect to all parametrizations, see e.g. condition (E) below.

3.2. **Maximal L^p-regularity.** Let $\mathcal{C}$ be a boundary differential operator of order $k=2$. Consider for a finite time interval $(0,T)$ the inhomogeneous Cauchy problem

$$\partial_t u + \mathcal{C}(t,x)u = g(t,x) \quad \text{on } (0,T)\times\Gamma, \qquad u|_{t=0} = u_0 \quad \text{on } \Gamma.$$

For $p\in(1,\infty)$ we take $g\in L^p((0,T)\times\Gamma)$ and are thus looking for solutions u that belong to the space

$$\mathbb{E}(\Gamma) := W^{1,p}(0,T;L^p(\Gamma)) \cap L^p(0,T;W^{2,p}(\Gamma)).$$

We want that for all parametrizations $\mathrm{g}: U\to\Gamma$ and all $|\gamma|\le 2$ the terms $c_\gamma^{\mathrm{g}} D_{n-1}^\gamma$ arising in the local representation of $\mathcal{C}$ are continuous from $\mathbb{E}(U)$ to $L^p((0,T)\times U)$. Then in particular $\mathcal{C}:\mathbb{E}(\Gamma)\to L^p((0,T)\times\Gamma)$ is continuous. Using Sobolev embeddings and Hölder's inequality, it can be shown as in [26, Lemma 1.3.15] that the following assumptions are sufficient for this purpose.

(R) Let $\mathrm{g}: U\to\Gamma$ be any parametrization of Γ. Then for $|\gamma|=2$ it holds $c_\gamma^{\mathrm{g}}\in BUC([0,T]\times U)$, and in case $|\gamma|<2$ one of the following conditions is valid: either $p>n+1$ and $c_\gamma^{\mathrm{g}}\in L^p((0,T)\times U)$, or there are $r_\gamma, s_\gamma\in[p,\infty)$ with $\frac{1}{s_\gamma}+\frac{n-1}{2r_\gamma}<1-\frac{|\gamma|}{2}$ such that $c_\gamma^{\mathrm{g}}\in L^{s_\gamma}(0,T;L^{r_\gamma}(U))$.

As structural conditions for $\mathcal{C}$ we assume *local parameter-ellipticity* (cf. [5] for the euclidian case). Observe that this is a condition only for the highest order coefficients.

(E) For all parametrizations $g: U\to\Gamma$, all $t\in[0,T]$, $x\in U$ and $\xi\in\mathbb{R}^{n-1}$ with $|\xi|=1$ it holds that $\sum_{|\gamma|=2} c_\gamma^{\mathrm{g}}(t,x)\xi^\gamma>0$.

Example 3.1. Let $\mathcal{C}u = -\mathrm{div}_\Gamma(\delta\nabla_\Gamma u)$ for a C^1-function $\delta:(0,T)\times\Gamma\to\mathbb{R}$ with $\delta\ge\delta_*>0$. Then $\mathcal{C}$ satisfies (E).

Our maximal L^p-regularity result is now as follows.

Theorem 3.2. *Let $p\in(1,\infty)$ and $T\in(0,\infty)$. Assume that $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma=\partial\Omega$, and that $\mathcal{C}$ is a differential operator on Γ satisfying* (R) *and* (E). *Then there is a unique solution*

$$u\in\mathbb{E}(\Gamma) = W^{1,p}(0,T;L^p(\Gamma))\cap L^p(0,T;W^{2,p}(\Gamma))$$

of the problem

$$\left\{\begin{array}{ll} \partial_t u + \mathcal{C}(t,x)u = g(t,x) & \text{on } (0,T)\times\Gamma, \\ u|_{t=0}=u_0 & \text{on } \Gamma, \end{array}\right. \tag{3.1}$$

if and only if the data is subject to

$$g\in L^p((0,T)\times\Gamma), \qquad u_0\in W^{2-2/p,p}(\Gamma).$$

Given $T_0>0$, there is a constant C, which is independent of the data and $T\in(0,T_0)$, such that

$$\|u\|_{\mathbb{E}(\Gamma)} \le C\left(\|g\|_{L^p((0,T)\times\Gamma)} + \|u_0\|_{W^{2-2/p,p}(\Gamma)}\right). \tag{3.2}$$

Moreover, in the autonomous case, i.e., if $\mathcal{C}$ is independent of t, then $-\mathcal{C}$ generates an analytic C_0-semigroup on $L^p(\Gamma)$.

Proof. *Step 1.* If $u\in\mathbb{E}(\Gamma)$ solves (3.1), then $g\in L^p((0,T)\times\Gamma)$ follows from (R), and further $u_0\in W^{2-2/p,p}(\Gamma)$ is a consequence of e.g. [27, Theorem 4.2]. Next, assume that a unique solution of (3.1) exists for all given data. Then the corresponding solution operator is continuous $L^p((0,T)\times\Gamma)\times W^{2-2/p,p}(\Gamma)\to\mathbb{E}(\Gamma)$

due to the open mapping theorem. This gives (3.2). The uniformity of the constant with respect to $T \in (0, T_0)$ follows from an extension-restriction argument and the uniqueness of solutions. Further, in this case the generator property of $-\mathcal{C}$ follows from [6, Corollary 4.4], and the strong continuity of the semigroup is a consequence of the density of $W^{2,p}(\Gamma)$ in $L^p(\Gamma)$.

We thus have to show the unique solvability of (3.1) in $\mathbb{E}(\Gamma)$ for all data $g \in L^p((0,T) \times \Gamma)$ and $u_0 \in W^{2-2/p,p}(\Gamma)$. A compactness argument shows that it suffices to do this for one (possibly small) $T > 0$, which is independent of the data.

Step 2. Choose a finite number of parametrizations g_i with domains U_i such that $\bigcup_i \mathrm{g}_i(U_i)$ covers Γ, and a partition of unity $\{\psi_i\}$ for Γ subordinate to this cover. Then $u \in \mathbb{E}(\Gamma)$ solves (3.1) if and only if for all i, the function $v_i := (\psi_i u) \circ \mathrm{g}_i$ solves

$$\partial_t v_i + \mathcal{C}^{\mathrm{g}_i} v_i = g_i, \quad \text{on } (0,T) \times U_i, \qquad v_i|_{t=0} = v_i^0, \quad \text{on } U_i. \tag{3.3}$$

Here the local operator $\mathcal{C}^{\mathrm{g}_i}$ is given by

$$\mathcal{C}^{\mathrm{g}_i}(t,x) = \sum_{|\gamma| \leq 2} c_\gamma^{\mathrm{g}_i}(t,x) D^\gamma, \qquad (t,x) \in (0,T) \times U_i,$$

and the transformed data is given by

$$g_i := (\psi_i g + [\mathcal{C}, \psi_i] u) \circ \mathrm{g}_i, \qquad v_i^0 := (\psi_i u^0) \circ \mathrm{g}_i,$$

where $[\cdot,\cdot]$ denotes the commutator bracket, i.e., $[\mathcal{C}, \psi_i] u = \mathcal{C}(\psi_i u) - \psi_i \mathcal{C} u$.

Identifying v_i, g_i and v_i^0 with their trivial extensions to $\mathbb{R}^{n-1}$, we obtain

$$v_i \in \mathbb{E}(\mathbb{R}^{n-1}), \qquad g_i \in L^p((0,T) \times \mathbb{R}^{n-1}), \qquad v_i^0 \in W^{2-2/p,p}(\mathbb{R}^{n-1}).$$

We extend the top order coefficients $c_\gamma^{\mathrm{g}_i}$, $|\gamma| = 2$, to coefficients $c_\gamma^i \in BUC([0,T] \times \mathbb{R}^{n-1})$, using only values from the image of $c_\gamma^{\mathrm{g}_i}$ (assuming e.g. U_i to be ball and reflecting on ∂U_i). By continuity of the $c_\gamma^{\mathrm{g}_i}$, the oscillation of the extend top order coefficients becomes small if the diameter of the U_i is small. The lower order coefficients $c_\gamma^{\mathrm{g}_i}$, $|\gamma| < 2$, are trivially extended to c_γ^i on $(0,T) \times \mathbb{R}^{n-1}$. The extended coefficients c_γ^i, $|\gamma| \leq 2$, induce a differential operator $\mathcal{C}^i(t,x)$ acting on functions over $(0,T) \times \mathbb{R}^{n-1}$, which satisfies (R) and (E) for $\Gamma = \mathbb{R}^{n-1}$. Note in particular that (E) is a pointwise condition. The (trivial extension of) v_i solves the full space problem

$$\partial_t w + \mathcal{C}^i(t,x) w = g_i \quad \text{on } (0,T) \times \mathbb{R}^{n-1}, \qquad w|_{t=0} = v_i^0 \quad \text{on } \mathbb{R}^{n-1}. \tag{3.4}$$

It is well-known that there is a solution operator $\mathcal{S}_i(g_i, v_0^i)$ for (3.4), which is continuous

$$\mathcal{S}_i : L^p((0,T) \times \mathbb{R}^{n-1}) \times W^{2-2/p,p}(\mathbb{R}^{n-1}) \to \mathbb{E}(\mathbb{R}^{n-1}).$$

We refer to [26, Proposition 2.3.2] for the case of top order coefficients with small oscillation, which applies to the present case. Hence

$$v_i = \mathcal{S}_i(g_i, v_i^0)|_{U_i}.$$

Step 3. Take $\phi_i \in C^\infty(\Gamma)$ with $\phi_i \equiv 1$ on $\operatorname{supp} \psi_i$ and $\operatorname{supp} \phi_i \subset \mathrm{g}_i(U_i)$. On the complete metric space

$$\mathbb{Y}_{u_0} := \{ u \in \mathbb{E}(\Gamma) \ : \ u|_{t=0} = u_0 \},$$

which is nonempty by [27, Lemma 4.3], we define the map $\mathcal{S}_{g,u_0}$ by

$$\mathcal{S}_{g,u_0}(u) := \sum_i \phi_i \mathcal{S}_i\big((\psi_i g + [\mathcal{C}, \psi_i] u) \circ \mathrm{g}_i, (\psi_i u_0) \circ \mathrm{g}_i \big)|_{U_i} \circ \mathrm{g}_i^{-1}.$$

Since $\mathcal{S}_i(g_i, v_i^0)|_{t=0} = v_i^0$, we have that $\mathcal{S}_{g,u^0}$ is a self-mapping on $\mathbb{Y}_{u_0}$. We show that it is a strict contraction if T is sufficiently small. For $u, v \in \mathbb{Y}_{u_0}$ we have

$$\begin{aligned} \|\mathcal{S}_{g,u^0}(u) - \mathcal{S}_{g,u^0}(v)\|_{\mathbb{E}(\Gamma)} &\leq C \sum_i \|\mathcal{S}_i\left([\mathcal{C}, \psi_i](u-v) \circ g_i, 0\right)\|_{\mathbb{E}(\mathbb{R}^{n-1})} \\ &\leq C \sum_i \|[\mathcal{C}, \psi_i](u-v) \circ g_i\|_{L^p((0,T)\times\mathbb{R}^{n-1})}. \end{aligned} \tag{3.5}$$

Since $[\mathcal{C}, \psi_i](u-v) \circ \mathrm{g}_i$ involves at most the first derivatives of $(u-v) \circ \mathrm{g}_i$, we have for arbitrary $\eta > 0$ that

$$\begin{aligned} \|[\mathcal{C}, \psi_i](u-v) \circ \mathrm{g}_i\|_{L^p((0,T)\times\mathbb{R}^{n-1})} &\leq C \, \|(u-v) \circ \mathrm{g}_i\|_{L^p(0,T;W^{1,p}(\mathbb{R}^{n-1}))} \\ &\leq \eta \|u-v\|_{L^p(0,T;W^{2,p}(\Gamma))} \\ &\quad + C_\eta \|u-v\|_{L^p((0,T)\times\mathbb{R}^{n-1})} \\ &\leq (\eta + C_\eta T)\|u-v\|_{\mathbb{E}(\Gamma)}, \end{aligned} \tag{3.6}$$

using the interpolation inequality for $W^{1,p}(\Gamma)$, Young's inequality and Poincare's inequality for $L^p(0,T;L^p(\Gamma))$ (see [27, Lemma 2.12]).

Thus if η and T are sufficiently small, then $\mathcal{S}_{g,u_0}$ has a unique fixed point on $\mathbb{Y}_{u_0}$. Observe that this is true for all g and u_0, and that the choice of T is independent of g and u_0. The considerations in Step 2 show that every solution of (3.1) is necessarily a fixed point of $\mathcal{S}_{g,u_0}$. We have thus already shown that solutions of (3.1) are unique. However, due to the nonempty intersections of the $\mathrm{g}_i(U_i)$, the fixed point does in general not solve (3.1).

Step 4. To find $g^* \in L^p((0,T)\times\Gamma)$ for which $\mathcal{S}_{g^*,u_0}$ solves (3.1) for given g and u_0 we consider the fixed point map $\mathcal{F}$, defined by

$$\mathcal{F} : L^p((0,T)\times\Gamma) \times W^{2-2/p,p}(\Gamma) \to \mathbb{E}(\Gamma), \qquad \mathcal{S}_{h,v_0}\left(\mathcal{F}(h,v_0)\right) = \mathcal{F}(h,v_0).$$

For $h \in L^p((0,T)\times\Gamma)$ and $u_0 \in W^{2-2/p,p}(\Gamma)$ we have $\mathcal{F}(h,u_0)|_{t=0} = u_0$ and

$$(\partial_t + \mathcal{C})\, \mathcal{F}(h,u_0) = h + \mathcal{K}h, \tag{3.7}$$

with the error term

$$\mathcal{K}h := \sum_i [\mathcal{C}, \phi_i]\mathcal{S}_i\big((\psi_i h + [\mathcal{C}, \psi_i]\mathcal{F}(h,u_0)) \circ \mathrm{g}_i, (\psi_i u_0) \circ \mathrm{g}_i\big)|_{U_i} \circ \mathrm{g}_i^{-1}.$$

We use again the contraction principle to show that the map $h \mapsto g - \mathcal{K}h$ has a fixed point g^* on $L^p((0,T)\times\Gamma)$. Then $\mathcal{F}(g^*, u_0)$ solves (3.1) for given $g \in L^p((0,T)\times\Gamma)$ by (3.7).

First note that $\mathcal{K}$ maps $L^p((0,T)\times\Gamma)$ into itself by construction. For $h_1, h_2 \in L^p((0,T)\times\Gamma)$ we have that $\mathcal{F}(h_1,v_0) - \mathcal{F}(h_2,v_0) = \mathcal{F}(h_1 - h_2, 0)$, since this difference is the unique solution of (3.1) with inhomogeneity $h_1 - h_2$ and trivial initial value. We thus obtain as above that

$$\begin{aligned} &\|\mathcal{K}h_1 - \mathcal{K}h_2\|_{L^p((0,T)\times\Gamma)} \\ &\quad \leq C \sum_i \|\mathcal{S}_i\big((\psi_i(h_1-h_2) + [\mathcal{C}, \psi_i]\mathcal{F}(h_1-h_2,0)) \circ \mathrm{g}_i, 0\big)\|_{L^p(0,T;W^{1,p}(\mathbb{R}^{n-1}))} \\ &\quad \leq (\eta + C_\eta T) \sum_i \|\mathcal{S}_i\big((\psi_i(h_1-h_2) + [\mathcal{C}, \psi_i]\mathcal{F}(h_1-h_2,0)) \circ \mathrm{g}_i, 0\big)\|_{\mathbb{E}(\Gamma)} \\ &\quad \leq C(\eta + C_\eta T)\left(\|h_1 - h_2\|_{L^p((0,T)\times\Gamma)} + \|[\mathcal{C}, \psi_i]\mathcal{F}(h_1-h_2,0)\|_{L^p((0,T)\times\Gamma)}\right), \end{aligned}$$

where $\eta > 0$ is arbitrary. Therefore $\mathcal{K}$ is a strict contraction for sufficiently small η and T if the second summand above satisfies

$$\|[\mathcal{C}, \psi_i]\mathcal{F}(h_1 - h_2, 0)\|_{L^p((0,T)\times\Gamma)} \leq C\,\|h_1 - h_2\|_{L^p((0,T)\times\Gamma)}, \tag{3.8}$$

with a constant C independent of T. To see this we estimate for $h \in L^p((0,T)\times\Gamma)$

$$\begin{aligned}\|\mathcal{F}(h,0)\|_{\mathbb{E}(\Gamma)} &= \|\mathcal{S}_{h,0}(\mathcal{F}(h,0))\|_{\mathbb{E}(\Gamma)} \\ &\leq \|\mathcal{S}_{h,0}(\mathcal{F}(h,0)) - \mathcal{S}_{h,0}(0)\|_{\mathbb{E}(\Gamma)} + \|\mathcal{S}_{h,0}(0)\|_{\mathbb{E}(\Gamma)} \\ &\leq (\varepsilon + C_\varepsilon T)\|\mathcal{F}(h,0)\|_{\mathbb{E}(\Gamma)} + C\|h\|_{L^p((0,T)\times\Gamma)}\end{aligned}$$

for given $\varepsilon > 0$ by (3.5) and (3.6). In this inequality, if ε and T are sufficiently small, then we may absorb $(\varepsilon + C_\varepsilon T)\|\mathcal{F}(h,0)\|_{\mathbb{E}(\Gamma)}$ into the left-hand side to obtain

$$\|\mathcal{F}(h,0)\|_{\mathbb{E}(\Gamma)} \leq C\|h\|_{L^p((0,T)\times\Gamma)}.$$

Now (3.8) follows from

$$\|[\mathcal{C}, \psi_i]\mathcal{F}(h_1 - h_2, 0)\|_{L^p((0,T)\times\Gamma)} \leq C\,\|\mathcal{F}(h_1 - h_2, 0)\|_{\mathbb{E}(\Gamma)},$$

which finishes the proof. □

4. Well-posedness and compactness of the solution semiflows

4.1. **Dirichlet problems.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. We assume that the operator $\mathcal{A}$ is given by

$$\mathcal{A}u = -\operatorname{div}\big(d\nabla u\big), \quad d \in C^\infty(\overline{\Omega}, \mathbb{R}), \quad d \geq d_* > 0, \tag{4.1}$$

where d_* is a constant. We first consider the linear inhomogeneous Dirichlet problem

$$\begin{cases} \lambda v + \mathcal{A}v = f & \text{in } \Omega, \\ v|_\Gamma = g & \text{on } \Gamma. \end{cases} \tag{4.2}$$

We have the following well-known a priori estimate of Agmon-Douglis-Nirenberg type. We denote $\operatorname{tr} u = u|_\Gamma$ for the trace on Γ.

Lemma 4.1. *Let $\mathcal{A}$ be given by (4.1). Given $p \in (1,\infty)$, there exists $\lambda_D \geq 0$ such that the following holds. For each $\lambda \geq \lambda_D$ the operator $(\lambda + \mathcal{A}, \operatorname{tr})$ extends to a continuous isomorphism*

$$(\lambda + \mathcal{A}, \operatorname{tr}) : H^{2\theta,p}(\Omega) \to L^p(\Omega) \times W^{2\theta-1/p,p}(\Gamma),$$

where $\theta \in (1/2p, 1]$, $\theta \neq 1/2 + 1/2p$. There are constants C_D (independent of λ) and C_λ such that $\mathcal{R}_\lambda := (\lambda + \mathcal{A}, \operatorname{tr})^{-1}$ satisfies

$$\|\mathcal{R}_\lambda(f,g)\|_{H^{2\theta,p}(\Omega)} \leq C_D \lambda^{-(1-\theta)}\|f\|_{L^p(\Omega)} + C_\lambda\|g\|_{W^{2\theta-1/p,p}(\Gamma)}.$$

Proof. The isomorphic properties of $(\lambda+\mathcal{A}, \operatorname{tr})$ for $\lambda \geq \lambda_D$ are proved in [8, Section 2]. Moreover, it is shown in [1, Theorem 12.2] that

$$\|\mathcal{R}_\lambda(f,0)\|_{H^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \qquad \|\mathcal{R}_\lambda(f,0)\|_{L^p(\Omega)} \leq C\lambda^{-1}\|f\|_{L^p(\Omega)},$$

for all $\lambda \geq \lambda_D$. Thus, by complex interpolation,

$$\|\mathcal{R}_\lambda(f,0)\|_{H^{2\theta,p}(\Omega)} \leq C_D\lambda^{-(1-\theta)}\|f\|_{L^p(\Omega)}.$$

Therefore

$$\begin{aligned}\|\mathcal{R}_\lambda(f,g)\|_{H^{2\theta,p}(\Omega)} &\leq \|\mathcal{R}_\lambda(f,0)\|_{H^{2\theta,p}(\Omega)} + \|\mathcal{R}_\lambda(0,g)\|_{H^{2\theta,p}(\Omega)} \\ &\leq C_D\lambda^{-(1-\theta)}\|f\|_{L^p(\Omega)} + C_\lambda\|g\|_{W^{2\theta-1/p,p}(\Gamma)},\end{aligned}$$

which proves the lemma. □

Remark 4.2.

(1) If $\theta < 1$, then $u = \mathcal{R}_\lambda(f, g)$ is a weak solution of (4.2).
(2) If $\mathcal{A} = -d\Delta$ for a constant $d > 0$, then one can take $\lambda_D = 0$.
(3) The value $\theta = 1/2 + 1/2p$ is only excluded to avoid the terminology of Besov spaces.

Let us now consider the nonlinear Dirichlet problem

$$\begin{cases} \lambda u + \mathcal{A}u = F(u) & \text{in } \Omega, \\ u|_\Gamma = u_\Gamma & \text{on } \Gamma. \end{cases} \tag{4.3}$$

Given $p \in (1, \infty)$ and $\theta \in (0, 1)$, for the nonlinearity we assume that

$$F : H^{2\theta,p}(\Omega) \to L^p(\Omega) \text{ is globally Lipschitzian with constant } c_F \geq 0. \tag{4.4}$$

Example 4.3. Let F be the superposition operator induced by a globally Lipschitzian $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, i.e., $F(u)(x) = f(u(x), \nabla u(x))$. Then F satisfies (4.4) for all p and $\theta \geq \frac{1}{2}$. If F is induced by a function that is not globally Lipschitzian, then (4.4) cannot hold.

Lemma 4.4. *Let $p \in (1, \infty)$, $\theta \in (1/2p, 1)$ with $\theta \neq 1/2 + 1/2p$ and assume* (4.1) *and* (4.4)*. Then there is $\lambda_* \geq \lambda_D$ such that for all $\lambda \geq \lambda_*$ the following holds. For all $u_\Gamma \in W^{2\theta-1/p,p}(\Gamma)$ there is a unique weak solution $u = \mathcal{D}_\lambda(u_\Gamma)$ of* (4.3)*, and the solution operator*

$$\mathcal{D}_\lambda : W^{2\theta-1/p,p}(\Gamma) \to H^{2\theta,p}(\Omega)$$

is globally Lipschitz continuous. If $F \in C^k(H^{2\theta,p}(\Omega), L^p(\Omega))$ for $k \in \mathbb{N} \cup \{\infty\}$, then $\mathcal{D}_\lambda \in C^k(W^{2\theta-1/p,p}(\Gamma), H^{2\theta,p}(\Omega))$.

Proof. Given $u_\Gamma \in W^{2\theta-1/p,p}(\Gamma)$, the function $u \in H^{2\theta,p}(\Omega)$ solves (4.3) if and only if

$$u = \mathcal{K}_\lambda(u, u_\Gamma) := \mathcal{R}_\lambda(F(u), u_\Gamma),$$

where $\mathcal{K}_\lambda : H^{2\theta,p}(\Omega) \times W^{2\theta-1/p,p}(\Gamma) \to H^{2\theta,p}(\Omega)$ by Lemma 4.1. For $u, v \in H^{2\theta,p}(\Omega)$ and fixed u_Γ we estimate, using Lemma 4.1 and (4.4),

$$\begin{aligned} \|\mathcal{K}_\lambda(u, u_\Gamma) - \mathcal{K}_\lambda(v, u_\Gamma)\|_{H^{2\theta,p}(\Omega)} &= \|\mathcal{R}_\lambda(F(u) - F(v), 0)\|_{H^{2\theta,p}(\Omega)} \\ &\leq C_D \lambda_*^{-(1-\theta)} c_F \|u - v\|_{H^{2\theta,p}(\Omega)}. \end{aligned}$$

Thus $\mathcal{K}_\lambda(\cdot, u_\Gamma)$ is a strict contraction on $H^{2\theta,p}(\Omega)$ for each u_Γ, provided λ_* is sufficiently large. The resulting unique fixed point is the unique solution of (4.3). The global Lipschitz continuity of the solution operator $\mathcal{D}_\lambda$ follows from

$$\begin{aligned} &\|\mathcal{D}_\lambda(u_\Gamma) - \mathcal{D}_\lambda(v_\Gamma)\|_{H^{2\theta,p}(\Omega)} \\ &\qquad = \|\mathcal{R}_\lambda(F(\mathcal{D}_\lambda(u_\Gamma)) - F(\mathcal{D}_\lambda(v_\Gamma)), u_\Gamma - v_\Gamma)\|_{H^{2\theta,p}(\Omega)} \\ &\qquad \leq C_D \lambda_*^{-(1-\theta)} c_F \|\mathcal{D}_\lambda(u_\Gamma) - \mathcal{D}_\lambda(v_\Gamma)\|_{H^{2\theta,p}(\Omega)} + C_\lambda \|u_\Gamma - v_\Gamma\|_{W^{2\theta-1/p,p}(\Gamma)}, \end{aligned}$$

and from $C_D \lambda_*^{-(1-\theta)} c_F < 1$. Finally, suppose that $F \in C^k$ and consider the map

$$\mathcal{F} : H^{2\theta,p}(\Omega) \times W^{2\theta-1/p,p}(\Gamma) \to H^{2\theta,p}(\Omega), \quad \mathcal{F}(u, u_\Gamma) = u - \mathcal{R}_\lambda(F(u), u_\Gamma).$$

The unique zero of $\mathcal{F}(\cdot, u_\Gamma)$ is $\mathcal{D}_\lambda(u_\Gamma)$. We have $\mathcal{F} \in C^k$, and for $h \in H^{2\theta,p}(\Omega)$ it holds $D_1\mathcal{F}(u, u_\Gamma)h = h - \mathcal{R}_\lambda(F'(u)h, 0)$. As above we can estimate

$$\|\mathcal{R}_\lambda(F'(u)h, 0)\|_{H^{2\theta,p}(\Omega)} \leq C_D \lambda_*^{-(1-\theta)} c_F \|h\|_{H^{2\theta,p}(\Omega)},$$

which yields that $D_1\mathcal{F}(u,u_\Gamma)$ is invertible for all (u,u_Γ) if λ_* is large. Thus $\mathcal{D}_\lambda \in C^k$ by the implicit function theorem. □

Remark 4.5. The number λ_* can be chosen such that $\lambda_* > \max\{\lambda_D, (C_D c_F)^{\frac{1}{1-\theta}}\}$. It tends to infinity as the global Lipschitz constant of F tends to infinity.

4.2. **Quasilinear boundary conditions of reactive-diffusive type.** We consider the following class of elliptic problems with quasilinear, nondegenerate dynamic boundary conditions of reactive-diffusive type,

$$\begin{cases} \lambda u + \mathcal{A}u = F(u) & \text{in } (0,T)\times\Omega, \\ \partial_t u_\Gamma + \mathcal{C}(u_\Gamma)u_\Gamma + \mathcal{B}(u) = G(u_\Gamma) & \text{on } (0,T)\times\Gamma, \\ u|_{t=0} = u_0 & \text{on } \Gamma. \end{cases} \tag{4.5}$$

This is a generalization of the prototype model (1.1) from the introduction.

We assume that $\mathcal{A}$ and F are as above, and that the nonlinear boundary differential operator $\mathcal{C}$ is given by

$$\mathcal{C}(u_\Gamma)v_\Gamma = -\mathrm{div}_\Gamma\big(\delta(\cdot,u_\Gamma)\nabla_\Gamma v_\Gamma\big), \quad \delta\in C^\infty(\Gamma\times\mathbb{R},\mathbb{R}), \quad \delta \geq \delta_* > 0, \tag{4.6}$$

where δ_* is a constant. The nonlinear map $\mathcal{B}$ couples the equations in the domain and on the boundary in a nontrivial way. Given $p\in(1,\infty)$, we assume that

$$\mathcal{B}: H^{2-1/p,p}(\Omega) \to L^p(\Gamma) \text{ is locally Lipschitzian.} \tag{4.7}$$

We do not impose any further structural condition for $\mathcal{B}$. In fact, it could vanish identically.

Example 4.6. The prototype for $\mathcal{B}$ is $\mathcal{B}(u) = B\nu\cdot(\nabla u)|_\Gamma$ for some $B\in C^\infty(\Gamma,\mathbb{R}^{n\times n})$, which satisfies (4.7) if $p>2$. For $B=\pm\mathrm{id}$ one obtains $\mathcal{B}=\pm\partial_\nu$. In the semilinear case one can also allow $p\leq 2$ for such $\mathcal{B}$, see Proposition 4.10 below.

Next, for the boundary nonlinearity G we assume

$$G: W^{2-2/p,p}(\Gamma)\to L^p(\Gamma) \text{ is locally Lipschitzian.} \tag{4.8}$$

Example 4.7. If $G(u_\Gamma)(x) = g(u_\Gamma(x))$ and $g:\mathbb{R}\to\mathbb{R}$ is locally Lipschitzian, then G satisfies (4.8) if $p>\frac{n+1}{2}$. If g depends in addition on $\nabla_\Gamma u_\Gamma$, then $p>n+1$ is required. These assertions are easily verified using Sobolev's embeddings. If g is polynomial, then the values of p can be lowered.

We have the following local well-posedness result for (4.5).

Theorem 4.8. *Let $p\in(n+1,\infty)$ and assume* (4.1), (4.4)*,* (4.6)*,* (4.7) *and* (4.8)*. Then there is λ_* such that for all $\lambda\geq\lambda_*$ the following holds true. The problem* (4.5) *generates a compact local semiflow of solutions on $W^{2-2/p,p}(\Gamma)$. For all $T\in(0,t^+(u_0))$ a solution $u=u(\cdot,u_0)$ enjoys the regularity*

$$u\in C([0,T];H^{2-1/p,p}(\Omega))\cap L^p(0,T;W^{2,p}(\Omega)),$$
$$u_\Gamma\in W^{1,p}(0,T;L^p(\Gamma))\cap L^p(0,T;W^{2,p}(\Gamma)).$$

Remark 4.9.

(1) The corresponding result for boundary conditions of purely reactive type, i.e., $\mathcal{C}\equiv 0$ and $\mathcal{B}=d\partial_\nu$, was shown in [7]. There the result is based on the generation properties of the Dirichlet-to-Neumann operator and thus requires a good sign of the normal derivative. Moreover, the solutions enjoy worse regularity properties up to $t=0$.

(2) Locally in time, the smoothing effect of the surface diffusion operator $\mathcal{C}$ is twofold. One has more regularity up to $t = 0$, and local well-posedness essentially becomes independent of the coupling $\mathcal{B}$, since it is of lower order with respect to $\mathcal{C}$. The latter was already observed in [37] for a linear problem in the special case $\mathcal{B} = -\partial_\nu$.

(3) The proof shows that one can take $\lambda_* = \max\{\lambda_D, (C_D c_F)^p\}$. If F is not globally Lipschitzian, then nonexistence, nonuniqueness and noncontinuation phenomena can occur (see [10]).

Proof of Theorem 4.8. Step 1. Let $\mathcal{D}_\lambda$ be the solution operator from Lemma 4.4 for the nonlinear Dirichlet problem (4.3), where $\theta = 1-1/p$. Observe that u satisfies (4.5) if and only if we have $u = \mathcal{D}_\lambda(u_\Gamma)$ and u_Γ solves

$$\left\{ \begin{array}{ll} \partial_t u_\Gamma + \mathcal{C}(u_\Gamma)u_\Gamma = G(u_\Gamma) - \mathcal{B}(\mathcal{D}_\lambda(u_\Gamma)) & \text{on } (0,T)\times\Gamma, \\ u_\Gamma|_{t=0} = u_0 & \text{on } \Gamma. \end{array} \right. \tag{4.9}$$

This is a quasilinear evolution equation for u_Γ. We verify the conditions of [21, Theorems 2.1 and 3.1, Corollary 3.2] to apply the abstract results on local well-posedness provided there.

By assumption and the Lipschitz continuity of $\mathcal{D}_\lambda$, the map $u_\Gamma \mapsto G(u_\Gamma) - \mathcal{B}(\mathcal{D}_\lambda(u_\Gamma))$ is locally Lipschitzian $W^{2-2/p,p}(\Gamma) \to L^p(\Gamma)$. Next we rewrite the leading term $\mathcal{C}(u_\Gamma)u_\Gamma$ into

$$\mathcal{C}(u_\Gamma)u_\Gamma = -\delta(\cdot,u_\Gamma)\Delta_\Gamma u_\Gamma - \nabla_\Gamma(\delta(\cdot,u_\Gamma))\nabla_\Gamma u.$$

For the first term we have

$$\|\delta(\cdot,u_\Gamma)\Delta_\Gamma w_\Gamma - \delta(\cdot,v_\Gamma)\Delta_\Gamma w_\Gamma\|_{L^p(\Gamma)} \le \|\delta(\cdot,u_\Gamma) - \delta(\cdot,v_\Gamma)\|_{L^\infty(\Gamma)}\|w_\Gamma\|_{W^{2,p}(\Gamma)}.$$

The superposition operator induced by δ is locally Lipschitzian as a map $C(\Gamma) \to C(\Gamma)$, and since $p > \frac{n+1}{2}$ we have that $W^{2-2/p,p}(\Gamma) \hookrightarrow C(\Gamma)$. Therefore $u_\Gamma \mapsto -\delta(\cdot,u_\Gamma)\Delta_\Gamma$ is locally Lipschitzian as a map $W^{2-2/p,p}(\Gamma) \to \mathcal{L}(W^{2,p}(\Gamma), L^p(\Gamma))$. Further, for each $u_\Gamma \in W^{2-2/p,p}(\Gamma)$ the function $-\delta(\cdot,u_\Gamma)$ belongs to $C(\Gamma)$. Hence by Theorem 3.2, the operator $-\delta(\cdot,u_\Gamma)\Delta_\Gamma$ with domain $W^{2,p}(\Gamma)$ on $L^p(\Gamma)$ enjoys the property of maximal L^p-regularity. Finally, if $p > n+1$ then $W^{2-2/p,p}(\Gamma) \hookrightarrow C^1(\Gamma)$, and this implies that $u_\Gamma \mapsto \nabla_\Gamma(\delta(\cdot,u_\Gamma))\nabla_\Gamma u$ is locally Lipschitzian $W^{2-2/p,p}(\Gamma) \to L^p(\Gamma)$. It thus follows from [21] that (4.9) generates a local solution semiflow as asserted.

Step 2. It remains to show the compactness of the solution semiflow. To this end we modify the arguments of [21, Section 3] appropriately. We will use the notion and properties of the weighted spaces L^p_μ used in [21], which are given by

$$L^p_\mu(0,T;E) := \left\{ v : (0,T) \to E \;:\; \|v\|^p_{L^p_\mu(0,T;E)} := \int_0^T t^{p(1-\mu)}|v(t)|^p_E \, dt < +\infty \right\},$$

for some $p \in (1,\infty)$, $\mu \in (1/p, 1]$, and a Banach space E with norm $|\cdot|_E$. The corresponding Sobolev spaces $W^{1,p}_\mu(0,T;E)$ are defined by

$$W^{1,p}_\mu(0,T;E) := \{v \in L^p_\mu(0,T;E) \,:\, \exists\, v' \in L^p_\mu(0,T;E)\}.$$

Let us now return to the proof. By assumption, it is easy to see that there exists a number $\mu \in (1/p, 1)$ with $2\mu - 2/p > 1 + \frac{n-1}{p}$. The same arguments as above show that $u_\Gamma \mapsto -\delta(\cdot,u_\Gamma)\Delta_\Gamma$ is locally Lipschitzian $W^{2\mu-2/p,p}(\Gamma) \to \mathcal{L}(W^{2,p}(\Gamma), L^p(\Gamma))$, and the lower order nonlinearities are locally Lipschitzian $W^{2\mu-2/p,p}(\Gamma) \to L^p(\Gamma)$.

Thus by [21, Theorem 2.1], for each $v_0 \in W^{2\mu-2/p,p}(\Gamma)$ there are $r, T > 0$ and a continuous map

$$\Phi : B_r(v_0) \subset W^{2\mu-2/p,p}(\Gamma) \to W^{1,p}_\mu(0,T; L^p(\Gamma)) \cap L^p_\mu(0,T; W^{2,p}(\Gamma))$$

such that $u_\Gamma = \Phi(u_0)$ solves (4.9) on $(0,T)$.

Now let M be a bounded subset of $W^{2-2/p,p}(\Gamma)$ such that $t^+(M) \geq T > 0$. Then M is relatively compact in $W^{2\mu-2/p,p}(\Gamma)$. Hence finitely many balls $B_{r_i} \subset W^{2\mu-2/p,p}(\Gamma)$ suffice to cover M, with corresponding solution maps Φ_i and times T_i as above. Let $T_0 = \min T_i$, and take $0 < t \leq T_0$. Then we have $u(t; M) = \bigcup_i \mathrm{tr}_t \Phi_i(B_i \cap M)$. By the continuity of Φ_i, the set $\Phi_i(B_i \cap M)$ is relatively compact in $W^{1,p}_\mu(0,T_0; L^p(\Gamma)) \cap L^p_\mu(0,T_0; W^{2,p}(\Gamma))$. Moreover, the trace tr_t at time t is continuous from the latter space into the higher regularity space $W^{2-2/p,p}(\Gamma)$, due to the fact that the weight $t^{p(1-\mu)}$ only has an effect at $t = 0$ (see [29, Proposition 3.1] and [27, Theorem 4.2]). Thus $u(t; M)$ is relatively compact in $W^{2-2/p,p}(\Gamma)$. Finally, in case $T_0 < t \leq T$ we obtain the relative compactness of $u(t; M)$ from $u(t; M) = u(t - T_0; u(T_0; M))$ and the continuity of $u(t - T_0; \cdot)$ on $W^{2-2/p,p}(\Gamma)$. □

Things are simpler in the semilinear case.

Proposition 4.10. *Let $p \in (1,\infty)$ and assume* (4.1) *and* (4.6)*, where δ is independent of u_Γ. Suppose that there is $\theta \in (1/2p, 1)$ such that*

$$F : H^{2\theta,p}(\Omega) \to L^p(\Omega), \quad G : W^{2\theta-1/p,p}(\Gamma) \to L^p(\Gamma), \quad \mathcal{B} : H^{2\theta,p}(\Omega) \to L^p(\Gamma),$$

where F is globally Lipschitzian and $G, \mathcal{B}$ are Lipschitzian on bounded sets. Then there is λ_* such that for all $\lambda \geq \lambda_*$ the following holds. For all $\sigma \in (\theta, 1)$ the problem (4.5) *generates a compact local semiflow of solutions on $W^{2\sigma-1/p,p}(\Gamma)$. A solution $u = u(\cdot; u_0)$ enjoys the regularity*

$$u \in C([0,t^+); H^{2\sigma,p}(\Omega)) \cap C(0,t^+; H^{2,p}(\Omega)),$$
$$u_\Gamma \in C([0,t^+); W^{2\sigma-1/p,p}(\Gamma)) \cap C^1(0,t^+; L^p(\Gamma)) \cap C(0,t^+; W^{2,p}(\Gamma)).$$

Proof. The equivalent formulation (4.9) of (4.5) is now an abstract semilinear problem. Since $W^{2,p}(\Gamma) \hookrightarrow L^p(\Gamma)$ is compact, the assertions follow from Theorem 3.2, Lemma 4.4 and e.g. [3, Theorems 2.1.1 and 3.2.1, Corollary 2.3.1]. □

4.3. **Compactness in the purely reactive case.** We complement the results in [7, 8] concerning compactness of the solution semiflow. We consider problems of type

$$\left\{\begin{array}{ll} \lambda u - \mathrm{div}(d\nabla u) = f(u) & \text{in } (0,T) \times \Omega, \\ \partial_t u_\Gamma + d\partial_\nu u = g(u_\Gamma) & \text{on } (0,T) \times \Gamma, \\ u|_{t=0} = u_0 & \text{on } \Gamma. \end{array}\right. \tag{4.10}$$

Throughout this section we assume that

$$d \in C^\infty(\overline{\Omega}), \qquad d \geq d_* > 0, \qquad f, g \in C^\infty(\mathbb{R}), \qquad |f'| \leq c_f. \tag{4.11}$$

The results of [7, Theorem 6.2] and [8, Theorem 2] can be summarized as follows.

Proposition 4.11. *Assume* (4.11)*. Then for $p \in (n,\infty)$ there is λ_* such that for all $\lambda \geq \lambda_*$ the problem* (4.10) *generates a local solution semiflow on $W^{1-1/p,p}(\Gamma)$. A solution u enjoys the regularity*

$$u \in C([0,t^+); W^{1,p}(\Omega)) \cap C^1(0,t^+; C^\infty(\Gamma)) \cap C(0,t^+; C^\infty(\overline{\Omega})).$$

For sufficiently large λ we define the Dirichlet-Neumann operator $\mathcal{N}_\lambda$ by

$$\mathcal{N}_\lambda u_\Gamma := d\partial_\nu \mathcal{R}_\lambda(0, u_\Gamma),$$

where $\mathcal{R}_\lambda$ is from Lemma 4.1, and consider it as an unbounded operator on $L^p(\Gamma)$ with domain $W^{1,p}(\Gamma)$. Using the solution operator $\mathcal{D}_\lambda$ from Lemma (4.4) for the inhomogeneous Dirichlet problem (4.3), we may rewrite (4.10) into the form

$$\left\{ \begin{array}{ll} \partial_t u_\Gamma + \mathcal{N}_\lambda u_\Gamma = g(u_\Gamma) - d\partial_\nu \mathcal{R}_\lambda(f(\mathcal{D}_\lambda(u_\Gamma)), 0) & \text{on } (0,T)\times\Gamma, \\ u_\Gamma|_{t=0} = u_0 & \text{on } \Gamma. \end{array} \right. \tag{4.12}$$

This is a nonlocal semilinear problem. The following generator properties of $\mathcal{N}_\lambda$ are shown in [7, Theorem 1.5] (see also [8, Theorem 3] and [9, Section 6]).

Proposition 4.12. *Let $d \in C^\infty(\overline{\Omega})$ with $d \geq d_* > 0$, and let λ be sufficiently large. Then for all $p \in (1,\infty)$ and $\theta \geq 0$ the operator $-\mathcal{N}_\lambda$ with domain $W^{\theta+1,p}(\Gamma)$ generates an analytic C_0-semigroup on $W^{\theta,p}(\Gamma)$.*

Now we prove the compactness of the semiflow generated by (4.10).

Proposition 4.13. *For each $p \in (n,\infty)$, the local solution semiflow from Proposition 4.11 is compact.*

Proof. Let $M \subset W^{1-1/p,p}(\Gamma)$ be bounded with $t^+(M) \geq T > 0$. Fix $t \in (0,T)$. Note that $D(\mathcal{N}_\lambda^{\alpha_2}) \hookrightarrow W^{s,p}(\Gamma) \hookrightarrow D(\mathcal{N}_\lambda^{\alpha_1})$ for $\alpha_2 > s > \alpha_1 \geq 0$. If α is sufficiently close to $1-1/p$, then Sobolev's embedding implies that $u_\Gamma \mapsto g(u_\Gamma)$ is Lipschitzian on bounded sets as a map $D(\mathcal{N}_\lambda^\alpha) \to L^p(\Gamma)$. Using the Lemmas 4.1 and 4.4 and the Lipschitz properties of f and $\mathcal{D}_\lambda$, for $u_\Gamma, v_\Gamma \in D(\mathcal{N}_\lambda^\alpha)$ and $\eta \in (0,\alpha)$ we estimate

$$\begin{aligned} \|d\partial_\nu \mathcal{R}_\lambda(f(\mathcal{D}_\lambda(u_\Gamma)) - f(\mathcal{D}_\lambda(v_\Gamma)), 0)\|_{L^p(\Gamma)} &\leq C\|f(\mathcal{D}_\lambda(u_\Gamma)) - f(\mathcal{D}_\lambda(v_\Gamma))\|_{L^p(\Omega)} \\ &\leq C\|u_\Gamma - v_\Gamma\|_{W^{\eta,p}(\Gamma)} \\ &\leq C\|u_\Gamma - v_\Gamma\|_{D(\mathcal{N}_\lambda^\alpha)}. \end{aligned}$$

Hence $u_\Gamma \mapsto d\partial_\nu \mathcal{R}_\lambda(f(\mathcal{D}_\lambda(u_\Gamma)), 0)$ is globally Lipschitzian as a map $D(\mathcal{N}_\lambda^\alpha) \to L^p(\Gamma)$. Therefore [3, Proposition 3.2.1] applies to (4.12), and we obtain that $u_\Gamma(t;M)$ is bounded in $D(\mathcal{N}_\lambda^\alpha)$ for all $\alpha \in (0,1)$. Since $W^{1,p}(\Gamma) \hookrightarrow L^p(\Gamma)$ is compact, we conclude that $u_\Gamma(t;M)$ is relatively compact in $W^{1-1/p,p}(\Gamma)$. □

5. Qualitative properties of classical solutions

In this section we study the qualitative properties of solutions of

$$\left\{ \begin{array}{ll} \lambda u - \operatorname{div}(d\nabla u) = f(u) & \text{in } (0,T)\times\Omega, \\ \partial_t u_\Gamma - \operatorname{div}_\Gamma(\delta\nabla_\Gamma u_\Gamma) + d\partial_\nu u = g(u_\Gamma) & \text{on } (0,T)\times\Gamma, \\ u|_{t=0} = u_0 & \text{on } \Gamma, \end{array} \right. \tag{5.1}$$

where we assume throughout that $\lambda \geq \lambda_*$ is sufficiently large (in dependence on the other parameters). We treat the two types of boundary conditions simultaneously and assume that

$$\left\{ \begin{array}{c} d \in C^\infty(\overline{\Omega}), \quad d \geq d_* > 0, \quad f, g \in C^\infty(\mathbb{R}), \quad |f'| \leq c_f, \quad p \in (n,\infty), \\ \delta \in C^\infty(\Gamma), \quad \text{and either } \delta \geq \delta_* > 0 \text{ or } \delta \equiv 0. \end{array} \right. \tag{5.2}$$

The local well-posedness of solutions is provided by the Propositions 4.10 and 4.11. To simplify the notation we set

$$\mathcal{X}_\delta := W^{2-2/p,p}(\Gamma) \quad \text{if } \delta \geq \delta_*, \qquad \mathcal{X}_\delta := W^{1-1/p,p}(\Gamma) \quad \text{if } \delta \equiv 0,$$

for the corresponding phase spaces. We will make essential use of the fact that for both types of boundary conditions the trace u_Γ of a solution of (5.1) satisfies

$$\left\{\begin{array}{ll} \partial_t u_\Gamma + \mathcal{C} u_\Gamma + \mathcal{N}_\lambda u_\Gamma = g(u_\Gamma) - d\partial_\nu \mathcal{R}_\lambda(f(\mathcal{D}_\lambda(u_\Gamma)), 0) & \text{on } (0,T)\times\Gamma, \\ u_\Gamma|_{t=0} = u_0 & \text{on } \Gamma, \end{array}\right. \tag{5.3}$$

where $\mathcal{C}u_\Gamma = -\mathrm{div}_\Gamma(\delta\nabla_\Gamma u_\Gamma)$, $\mathcal{N}_\lambda$ is the Dirichlet-Neumann operator and $\mathcal{R}_\lambda$ is from Lemma 4.1. By Theorem 3.2 and Proposition 4.12, the operators $\mathcal{C}$ and $\mathcal{N}_\lambda$ are both the negative generator of an analytic C_0-semigroup on $L^p(\Gamma)$. Since $\mathcal{N}_\lambda$ is of lower order with respect to $\mathcal{C}$, the same is true for $\mathcal{C}+\mathcal{N}_\lambda$. Therefore we may represent u_Γ by the variation of constants formula with an inhomogeneous term as above.

5.1. **Classical solutions.** We show the smoothness of solutions in space and time. Besides its own interest, this will become important to apply the comparison result Lemma 5.3 below and to show that (5.1) is of gradient structure (see Section 5.4).

The key to smoothness in time is the following.

Lemma 5.1. *Suppose that* (5.2) *holds, and that* $\varphi \in C^\infty(0,T;W^{1-1/p,p}(\Gamma))$. *For each* $t\in(0,T)$, *denote by* $u = u(t,\cdot)$ *the unique weak solution of*

$$\left\{\begin{array}{ll} \lambda u - \mathrm{div}(d\nabla u) = f(u) & \text{in } \Omega, \\ u|_\Gamma = \varphi(t) & \text{on } \Gamma. \end{array}\right. \tag{5.4}$$

Then $u \in C^\infty(0,T;H^{1,p}(\Omega))$.

Proof. Define $\mathcal{F} : (0,T)\times H^{1,p}(\Omega) \to H^{1,p}(\Omega)$ by

$$\mathcal{F}(t,v) := v - \mathcal{R}_\lambda\big(f(v),\varphi(t)\big).$$

By Lemma 4.4, for each t the unique zero of $\mathcal{F}$ is $u(t,\cdot)$. The assumption on p guarantees that the superposition operator $v\mapsto f(v)$ belongs to $C^\infty(H^{1,p}(\Omega),L^p(\Omega))$, with derivative $h\mapsto f'(v)h$. The regularity of φ and the continuity of $\mathcal{R}_\lambda$ thus show that $\mathcal{F}\in C^\infty$. At $v\in H^{1,p}(\Omega)$ the derivative $D_2\mathcal{F}(t,v)$ is given by $h\mapsto h-\mathcal{R}_\lambda(f'(v)h,0)$, and it holds

$$\|\mathcal{R}_\lambda(f'(v)h,0)\|_{H^{1,p}(\Omega)} \le C_D\lambda_*^{-1/2}c_f\|h\|_{H^{1,p}(\Omega)}.$$

Therefore $D_2\mathcal{F}(t,v)$ is invertible for all t and all v. We obtain that for every $t_0\in(0,T)$ there are $\varepsilon>0$ and a function $\Phi\in C^\infty\big(t_0-\varepsilon,t_0+\varepsilon;H^{1,p}(\Omega)\big)$ such that $\Phi(t_0)=u(t_0,\cdot)$. By uniqueness one has in fact that $\Phi(t)=u(t,\cdot)$ for all $t\in(t_0-\varepsilon,t_0+\varepsilon)$. Hence $u\in C^\infty\big(0,T;H^{1,p}(\Omega)\big)$ as asserted. ☐

After this preparation we can show the smoothness of solutions.

Proposition 5.2. *Let* (5.2) *hold. Then for all* $u_0\in\mathcal{X}_\delta$ *the solution* u *of* (5.1) *satisfies* $u\in C^\infty((0,t^+)\times\overline{\Omega})$.

Proof. Throughout we fix $T<t^+$.

Step 1. First let $\delta\ge\delta_*$. For sufficiently large ρ the operator $\rho+\mathcal{C}$ is invertible and commutes with $-\mathcal{C}$. Employing local arguments as in the proof of Theorem (3.2) and interpolation, we obtain that $\rho+\mathcal{C}$ is an isomorphism $W^{2+\theta,p}(\Gamma)\to W^{\theta,p}(\Gamma)$ for all $\theta\ge0$. Thus $-\mathcal{C}$ with domain $W^{2+\theta,p}(\Gamma)$ generates an analytic C_0-semigroup on $W^{\theta,p}(\Gamma)$ for all θ.

Step 2. The trace u_Γ may be represented by

$$u_\Gamma(t,\cdot) = e^{-\mathcal{C}t}u_0 + e^{-\mathcal{C}\cdot} * \big(g(u_\Gamma) - d\partial_\nu u\big)(t), \quad t \in (0,T]. \tag{5.5}$$

Since $u \in C([0,T];H^{2-1/p}(\Omega))$ we have $g(u_\Gamma) - d\partial_\nu u \in C([0,T];W^{1-2/p,p}(\Gamma))$. We may thus consider (5.5) as an identity on $W^{1-2/p,p}(\Gamma)$, and obtain from [25, Corollary 4.3.9] that

$$u_\Gamma \in C^1((0,T];W^{1-2/p,p}(\Gamma)) \cap C((0,T];W^{3-2/p,p}(\Gamma)).$$

Since $u = \mathcal{R}_\lambda(f(u),u_\Gamma)$ and $f(u(t,\cdot)) \in H^{1-1/p}(\Omega)$, we further obtain from [1, Theorem 13.1] that $u(t,\cdot) \in H^{3-1/p,p}(\Omega)$ for all t, and that

$$\begin{aligned} \|u(t_1,\cdot) - u(t_2,\cdot)\|_{H^{3-1/p,p}(\Omega)} &\le C\big(\|f(u(t_1,\cdot)) - f(u(t_2,\cdot))\|_{H^{1-1/p}(\Omega)} \\ &\qquad + \|u_\Gamma(t_1,\cdot) - u_\Gamma(t_2,\cdot)\|_{W^{3-2/p,p}(\Gamma)}\big) \end{aligned} \tag{5.6}$$

with a constant C independent of $t_1,t_2 \in (0,T]$. Thus $u \in C((0,T];W^{3-1/p,p}(\Omega))$. An iteration of these arguments together with Sobolev's embeddings gives

$$u \in C^1((0,T];C^\infty(\Gamma)) \cap C((0,T];C^\infty(\overline{\Omega})).$$

Now it follows from (5.5) that $u_\Gamma \in C^\infty((0,T];C^\infty(\Gamma))$. Moreover, Lemma 5.1 implies that $u \in C^\infty((0,T];H^{1,p}(\Omega))$.

Step 3. Let now $\delta \equiv 0$. By Proposition 4.11 we have $u \in C^1((0,T];C^\infty(\Gamma)) \cap C((0,T];C^\infty(\overline{\Omega}))$, and further

$$u_\Gamma(t,\cdot) = e^{-\mathcal{N}_\lambda t}u_0 + e^{-\mathcal{N}_\lambda \cdot} * \big(g(u_\Gamma) - d\partial_\nu \mathcal{R}_\lambda(f(u),0)\big)(t), \quad t \in (0,T].$$

As above, this formula yields $u_\Gamma \in C^\infty((0,T]\times\Gamma)$ and then $u \in C^\infty((0,T];H^{1,p}(\Omega))$ by Lemma 5.1.

Step 4. For both types of boundary conditions it now follows from the linearity and the continuity of $\mathcal{R}_\lambda$ that

$$\partial_t^k u = \mathcal{R}_\lambda(\partial_t^k(f(u)), \partial_t^k u_\Gamma)$$

for all $k \in \mathbb{N}$. Now argue by induction and suppose that $\partial_t^{k-1}u \in C((0,T];C^\infty(\overline{\Omega}))$. Note that $\partial_t^k(f(u))$ is of the form $f'(u)\partial_t^k u + \psi$, where $\psi \in C((0,T];C^\infty(\overline{\Omega}))$ is a polynomial in the derivatives of u up to the order $k-1$ and derivatives of f with u inserted. Since $|f'(u)| \le c_f$ we may apply [1, Theorem 13.1] to $\mathcal{A} - f'(u)$ for all $\lambda \ge \lambda_D + c_f$ and estimate as in (5.6) to obtain $\partial_t^k u \in C((0,T];C^\infty(\overline{\Omega}))$. $\square$

5.2. **Blow-up.** In this subsection we assume that $d \equiv d_* > 0$ and $\delta \equiv \delta_* \ge 0$ are constants. Our blow-up results are based on the method of subsolutions and the following comparison lemma. Its proof is inspired by [31, Theorem II.3].

Lemma 5.3. *Assume $f,g \in C^1(\mathbb{R})$ with $|f'| \le c_f$, $\lambda \ge c_f$, $d > 0$ and $\delta \ge 0$. If*

$$u,v \in C([0,T]\times\overline{\Omega}) \cap C^1((0,T];C(\Gamma)) \cap C((0,T];C^2(\overline{\Omega}))$$

satisfy

$$\begin{cases} \lambda v - d\Delta v - f(v) \ge \lambda u - d\Delta u - f(u) & \text{in } (0,T]\times\Omega, \\ \partial_t v_\Gamma - \delta\Delta_\Gamma v_\Gamma + d\partial_\nu v - g(v_\Gamma) \ge \partial_t u_\Gamma - \delta\Delta_\Gamma u_\Gamma + d\partial_\nu u - g(u_\Gamma) & \text{on } (0,T]\times\Gamma, \\ v|_{t=0} \ge u|_{t=0} & \text{on } \Gamma, \end{cases}$$

then $v \ge u$ on $[0,T]\times\overline{\Omega}$.

Proof. The assumptions on f and λ imply that the function

$$a(t,x) = \frac{\lambda v(t,x) - f(v(t,x)) - (\lambda u(t,x) - f(u(t,x)))}{v(t,x) - u(t,x)}$$

is continuous and nonnegative on $[0,T]\times\overline{\Omega}$. Moreover, for all $(t,x)\in[0,T]\times\overline{\Omega}$ we can write

$$g(v(t,x)) - g(u(t,x)) = (L - b(t,x))(v(t,x) - u(t,x)),$$

where $L > 0$ is a constant and b is continuous and nonnegative on $[0,T]\times\Gamma$. Define

$$w(t,x) := e^{Lt}(v(t,x) - u(t,x)).$$

We suppose that $m := \min_{[0,T]\times\overline{\Omega}} w < 0$ and derive a contradiction. Let $(t_0,x_0)\in(0,T]\times\overline{\Omega}$ be such that $m = w(t_0,x_0)$. The function w satisfies

$$\lambda w - d\Delta w \geq e^{Lt_0}(f(v) - f(u)) \qquad \text{in } \{t_0\}\times\Omega,$$

and is thus a classical solution of

$$d\Delta w - e^{Lt_0}(\lambda v - f(v) - (\lambda u - f(u))) = d\Delta w - aw \leq 0 \qquad \text{in } \{t_0\}\times\Omega.$$

Since $-a \leq 0$ we deduce from the strong maximum principle [16, Theorem 3.5] that $x_0\in\Gamma$. Now the Hopf lemma [16, Lemma 3.4] implies $\partial_\nu w(t_0,x_0) < 0$. Therefore

$$\partial_t w(t_0,x_0) - \delta\Delta_\Gamma w(t_0,x_0) + b(t_0,x_0)w(t_0,x_0) > 0. \tag{5.7}$$

As $b\geq 0$ we have $b(t_0,x_0)w(t_0,x_0)\leq 0$, and further $\partial_t w(t_0,x_0)\leq 0$ since $t\mapsto w(t,x_0)$ attains its minimum in t_0. Moreover, in case $\delta > 0$, take orthogonal coordinates $\mathrm{g} : U\subset\mathbb{R}^{n-1}\to\Gamma$ for $x_0\in\Gamma$, with $\mathrm{g}(y_0) = x_0$ for some $y_0\in U$. Then $y\mapsto w(t_0,\mathrm{g}(y))$ has a local minimum in y_0, which implies that $\nabla_y w(t_0,\mathrm{g}(y_0)) = 0$ and $\Delta_y w(t_0,\mathrm{g}(y_0))\geq 0$. Hence the formula for Δ_Γ in coordinates yields

$$\Delta_\Gamma w(t_0,x_0) = \Delta_\Gamma w(t_0,\mathrm{g}(y_0)) = \Delta_y w(t_0,\mathrm{g}(y_0)) \geq 0.$$

The signs of the terms on the left-hand side of (5.7) lead to a contradiction. □

To obtain appropriate subsolutions we modify the ones from [2, Lemma 4.1].

Proposition 5.4. *Let* (5.2) *hold and assume* $d\equiv d_*$ *and* $\delta\equiv\delta_*$. *Let further*

$$\frac{g(\xi)}{\xi^q}\to+\infty \qquad \text{as } \xi\to+\infty,$$

for some $q > 1$. *Then there is* $C > 0$ *such that if* $u_0\in\mathcal{X}_\delta$ *satisfies* $u_0\geq C$, *then the solution of* (5.1) *blows up in finite time.*

Remark 5.5. For $\delta = 0$, blow-up results for (5.1) with $f\neq 0$ were obtained in [39] by the so-called concavity method.

Proof. For $1 < r \leq q$, let $\varphi(s) := (c - (r-1)s)^{-1/(r-1)}$, where $c := (r-1)(\max_{y\in\overline{\Omega}}\sum_i y_i + 1)$, such that $\varphi' = \varphi^r$ and $\varphi'' = r\varphi^{2r-1}$. Define $\underline{u}$ by

$$\underline{u}(t,x) := \varphi\Big(\sum_i x_i + t\Big) = \Big((r-1)\Big[\max_{y\in\overline{\Omega}}\sum_i y_i - \sum_i x_i + 1 - t\Big]\Big)^{-1/(r-1)},$$

which is well-defined on $\overline{\Omega}$ as long as $t < 1$. Observe that $\underline{u}$ is positive and that

$$\text{for all } K > 0 \text{ there is } r > 1 \text{ such that} \qquad \underline{u}(t,x)\geq K \quad \text{on } [0,1)\times\overline{\Omega}. \tag{5.8}$$

We check that $\underline{u}$ is a subsolution of (5.1) on $(0,1)\times\overline{\Omega}$ for a suitable choice of r.

First consider the elliptic equation. The assumption on λ and f yields

$$\lambda \underline{u} - f(\underline{u}) \leq (\lambda + c_f)\underline{u} - f(0),$$

and we have $\Delta \underline{u} = nr\underline{u}^{2r-1}$. By (5.8) (with $\underline{u}^{2(r-1)}$ instead of $\underline{u}$) we can achieve the inequality

$$(\lambda + c_f) \leq dnr\underline{u}^{2(r-1)} + f(0)/\underline{u} \qquad \text{on } (0,1)\times\Omega$$

if r is sufficiently close to 1.

For the boundary equation we have $\partial_t \underline{u} = \underline{u}^r$ and $\partial_\nu \underline{u} = (\nu \cdot \mathbf{1})\underline{u}^r$, where $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^n$. To treat the Laplace-Beltrami term in case $\delta > 0$, fix $x_0 \in \Gamma$ and take orthogonal coordinates $\mathrm{g} : U \subset \mathbb{R}^{n-1} \to \Gamma$ for x_0, such that $x_0 = \mathrm{g}(y_0)$ for some $y_0 \in U$. Let $|\mathrm{G}|$ be the Gramian and let $\mathrm{G}^{-1} = (\mathrm{g}^{ij})_{i,j=1,\dots,n-1}$ be the inverse fundamental form with respect to g. We write $a(x) = \sum_i x_i$ for simplicity. Since $(\mathrm{g}^{ij})_{y=y_0}$ equals the Kronecker symbol, we have

$$\begin{aligned}(\Delta_\Gamma \underline{u})(t, x_0) &= \sum_{i,j=1}^{n-1} \partial_i \big[\sqrt{|\mathrm{G}|}\mathrm{g}^{ij}\partial_j(\varphi(a\circ \mathrm{g}(y_0) + t))\big] \\ &= \underline{u}^r(t, x_0)\Delta_\Gamma a(x_0) + r\underline{u}^{2r-1}\sum_{i=1}^{n-1} |\partial_i(a \circ \mathrm{g})(y_0)|^2 \\ &\geq m\underline{u}^r(t, x_0),\end{aligned}$$

where $m = \min_{x\in\Gamma} \Delta_\Gamma a(x)$. Therefore on $(0,1)\times\Gamma$ we have

$$\partial_t \underline{u} - \delta\Delta_\Gamma \underline{u} + d\partial_\nu \underline{u} \leq (1 - \delta m + \nu\cdot\mathbf{1})\underline{u}^r \leq g(\underline{u})$$

when choosing r such that $(1-\delta m+\nu\cdot\mathbf{1}) \leq g(\underline{u})/\underline{u}^r$ on $(0,1)\times\Gamma$, which is possible by assumption on g and (5.8). Hence $\underline{u}$ is a subsolution of (5.1) if r is appropriate.

Now take $u_0 \in \mathcal{X}_\delta$ with $u_0 \geq \underline{u}|_{t=0}$ on $\overline{\Omega}$. Let u be the corresponding classical solution of (5.1). Then $u \geq \underline{u}$ on $\overline{\Omega}$ by Lemma 5.3, as long as u exists. Thus u blows up at $t = 1$. □

In case $f \equiv 0$ we can refine the blow-up condition for g.

Proposition 5.6. *Let $d > 0$ and $\delta \geq 0$. Suppose that there is ξ_0 such that $g(\xi) > 0$ for $\xi \geq \xi_0$, and that*

$$\int_{\xi_0}^{\infty} \frac{d\xi}{g(\xi)} < \infty.$$

Then there is $C > 0$ such that for all $u_0 \in \mathcal{X}_\delta$ the solution of

$$\left\{\begin{array}{ll} \Delta u = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t u_\Gamma - \delta\Delta_\Gamma u_\Gamma + d\partial_\nu u = g(u_\Gamma) & \text{on } (0,T)\times\Gamma, \\ u|_{t=0} = u_0 & \text{on } \Gamma, \end{array}\right. \tag{5.9}$$

blows up in finite time.

Remark 5.7. Under the additional assumption that g is entirely positive, the above result was shown in [20, Theorem 1] for $\delta = 0$.

Proof. For a constant initial value $\underline{u}_0 > 0$ the solution of (5.9) is given by the solution $\underline{u}$ of $u' = g(u)$ with $u|_{t=0} = \underline{u}_0$. If $\underline{u}_0$ is sufficiently large, then it is well-known that the condition on g implies that $\underline{u}$ blows up in finite time. By Lemma 5.3, any solution of (5.9) with initial value $u_0 \geq \underline{u}_0$ blows up as well. □

5.3. **Global existence.** We now return to the slightly more general assumptions (5.2) with variable diffusion coefficients. First we refine the blow-up conditions and show that for both types of boundary conditions an L^∞-bound for u_Γ suffices for global existence.

Lemma 5.8. *Let* (5.2) *hold, and assume that for $u_0 \in \mathcal{X}_\delta$ the solution u of* (5.1) *satisfies*

$$u_\Gamma \in L^\infty((0,t^+) \times \Gamma).$$

Then $t^+ = \infty$.

Proof. Suppose $t^+ < \infty$. We show $u_\Gamma \in L^\infty(0,t^+;\mathcal{X}_\delta)$ to derive a contradiction. In both cases $\delta \geq \delta_*$ and $\delta \equiv 0$, for $T < t^+$ we may use the variation of constants formula to estimate as in the proof of [3, Proposition 3.2.1],

$$\begin{aligned} \sup_{t\in[0,T]} \|u_\Gamma(t)\|_{\mathcal{X}_\delta} \leq C_{t^+}\big(1 + &\sup_{t\in[0,T]} \|g(u_\Gamma(t))\|_{L^p(\Gamma)} \\ &+ \sup_{t\in[0,T]} \|d\partial_\nu \mathcal{R}_\lambda(f(\mathcal{D}_\lambda(u_\Gamma(t))),0)\|_{L^p(\Gamma)}\big). \end{aligned} \tag{5.10}$$

By assumption, the second summand is bounded independent of $T < t^+$. For the third summand we have by Lemma 4.4 that

$$\|d\partial_\nu \mathcal{R}_\lambda(f(\mathcal{D}_\lambda(u_\Gamma(t))),0)\|_{L^p(\Gamma)} \leq C\|f(\mathcal{D}_\lambda(u_\Gamma(t)))\|_{L^p(\Omega)} \leq C_\eta \|u_\Gamma(t)\|_{W^{\eta,p}(\Gamma)},$$

where $\eta > 0$ is small. Given $\varepsilon > 0$, it follows from the interpolation inequality and Young's inequality that

$$\|u_\Gamma(t)\|_{W^{\eta,p}(\Gamma)} \leq \varepsilon \|u_\Gamma(t)\|_{\mathcal{X}_\delta} + C_\varepsilon \|u_\Gamma(t)\|_{L^p(\Gamma)} \leq \varepsilon \sup_{t\in[0,T]} \|u_\Gamma(t)\|_{\mathcal{X}_\delta} + C_\varepsilon.$$

For sufficiently small ε we may absorb $\varepsilon \sup_{t\in[0,T]} \|u_\Gamma(t)\|_{\mathcal{X}_\delta}$ into the left-hand side of (5.10). We thus find a bound for $\sup_{t\in[0,T]} \|u_\Gamma(t)\|_{\mathcal{X}_\delta}$ that is independent of $T < t^+$. Hence $u_\Gamma \in L^\infty(0,t^+;\mathcal{X}_\delta)$. □

Remark 5.9. It follows from the proof above that if g grows asymptotically at most polynomial, then $u_\Gamma \in L^\infty(0,t^+;L^q(\Gamma))$ for sufficiently large $q < \infty$ is already sufficient for global existence.

Before continuing we need the following inequality of Poincaré-Young type.

Lemma 5.10. *For all $p \in (1,\infty)$ and $\varepsilon \in (0,1)$ there is $\tau > 0$ such that*

$$\|u\|_{L^p(\Gamma)} \leq \varepsilon \|\nabla u\|_{L^p(\Omega)} + \varepsilon^{-\tau} \|u\|_{L^1(\Gamma)}, \qquad \textit{for all } u \in W^{1,p}(\Omega). \tag{5.11}$$

Proof. Step 1. We use the Poincaré inequality proved in [30, Lemma 3.1] to estimate

$$\|u\|_{L^p(\Omega)} \leq \|u - \frac{1}{|\Gamma|}\int_\Gamma u\|_{L^p(\Omega)} + C\|u\|_{L^p(\Gamma)} \leq C\big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Gamma)}\big).$$

Thus $\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Gamma)}$ is an equivalent norm on $W^{1,p}(\Omega)$.

Step 2. By a scaling argument it suffices to prove the inequality for $\|u\|_{L^p(\Gamma)} = 1$. Suppose that there is no $\tau > 0$ such that the inequality holds for a given $\varepsilon \in (0,1)$. Then for any $k \in \mathbb{N}$ there is $u_k \in W^{1,p}(\Omega)$ such that

$$\|u_k\|_{L^p(\Gamma)} = 1 \geq \varepsilon \|\nabla u_k\|_{L^p(\Omega)} + \varepsilon^{-k}\|u_k\|_{L^1(\Gamma)}.$$

It follows from this inequality and Step 1 that the resulting sequence (u_k) is bounded in $W^{1,p}(\Omega)$. Since the trace operator is a compact map from $W^{1,p}(\Omega)$ into $L^p(\Gamma)$ and

into $L^1(\Gamma)$, we find a subsequence, again denoted by (u_k), that converges in $L^p(\Gamma)$ and in $L^1(\Gamma)$ to some limit u. By assumption we have $\|u\|_{L^p(\Gamma)} = 1$. On the other hand, the inequality shows that $\|u_k\|_{L^1(\Gamma)} \le \varepsilon^k$ for all k, such that $\|u\|_{L^1(\Gamma)} = 0$ and thus $u|_\Gamma = 0$. This is a contradiction. $\square$

We verify an $L^\infty(\Gamma)$-bound for solutions of (5.1) under the assumption that

$$g(\xi)\xi \le c_g(\xi^2 + 1) \quad \text{for all } \xi \in \mathbb{R}, \tag{5.12}$$

where c_g is a nonnegative constant. Observe that this sign condition complements the sufficient condition from Proposition 5.4 for blow-up.

Proposition 5.11. *Let* (5.2) *hold, and assume* (5.12). *Then for all $u_0 \in \mathcal{X}_\delta$ the classical solution of* (5.1) *exists globally in time, i.e., $t^+ = \infty$.*

Proof. We suppose that $t^+ < \infty$ and show $u_\Gamma \in L^\infty((0, t^+) \times \Gamma)$ to derive a contradiction to Lemma 5.8.

Step 1. Let $T < t^+$. By an iteration argument we will first show that

$$\|u_\Gamma\|_{L^\infty((0,T)\times\Gamma)} \le C \max\left(\|u_0\|_{L^\infty(\Gamma)}, \|u_\Gamma\|_{L^\infty(0,T;L^2(\Gamma))}\right), \tag{5.13}$$

where C is independent of u_Γ and T. Let $k \in \mathbb{N}$, fix $t \in (0,T)$ and write $u = u_\Gamma = u(t,\cdot)$. We multiply the equation on Γ by u^{2^k-1} and integrate by parts on Γ to obtain

$$\begin{aligned}\frac{d}{dt}\int_\Gamma u^{2^k}\,dS &= -(2^k-1)2^{2-k}\int_\Gamma \delta|\nabla_\Gamma\left(u^{2^{k-1}}\right)|^2 dS\\ &\quad + 2^k\int_\Gamma g(u)u^{2^k-1}dS - 2^k\int_\Gamma d\partial_\nu u u^{2^k-1}dS.\end{aligned}$$

Multiplying the equation on Ω by u^{2^k-1} gives

$$\begin{aligned}-2^k d\int_\Gamma \partial_\nu u u^{2^k-1}dS &= -(2^k-1)2^{2-k}\int_\Omega d|\nabla\left(u^{2^{k-1}}\right)|^2 dx\\ &\quad + 2^k\int_\Omega\left(f(u)u^{2^k-1} - \lambda u^{2^k}\right)dx.\end{aligned}$$

Using $-(2^k-1)2^{2-k} \le -2$, that f is globally Lipschitzian and that $\lambda \ge c_f$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Gamma u^{2^k}\,dS &\le -2d_*\int_\Omega |\nabla\left(u^{2^{k-1}}\right)|^2 dx\\ &\quad + 2^k\int_\Omega\left(f(u)u^{2^k-1} - \lambda u^{2^k}\right)dx + 2^k\int_\Gamma g(u)u^{2^k-1}dS\\ &\le -2d_*\int_\Omega |\nabla\left(u^{2^{k-1}}\right)|^2 dx + C\,2^k\int_\Gamma u^{2^k}\,dS + C\,2^k.\end{aligned} \tag{5.14}$$

Given $\varepsilon > 0$, it follows from Lemma 5.10 that there is $\tau > 1$ such that

$$-\int_\Omega |\nabla v|^2\,dx \le -\varepsilon^{-1}\int_\Gamma v^2\,dS + \varepsilon^{-\tau}\left(\int_\Gamma |v|\,dS\right)^2.$$

Choosing $\varepsilon = \delta 2^{-k}$ with sufficiently small $\delta > 0$, we obtain that

$$\frac{d}{dt}\int_\Gamma u^{2^k}\,dS \le -2^k\int_\Gamma u^{2^k}\,dS + C2^{k\tau}\int_\Gamma u^{2^{k-1}}\,dS + C2^k, \qquad k \in \mathbb{N}. \tag{5.15}$$

Now (5.13) follows from a standard Moser-Alikakos iteration procedure as presented e.g. in [3, Proposition 9.3.1] (see also [26, Lemma 5.5.3]).

Step 2. Set $\varphi = \|u_\Gamma\|^2_{L^2(\Gamma)}$. Employing (5.14) with $k = 1$, we get $\varphi' \leq C_1\varphi + C_2$, which we can integrate to

$$\varphi(t) \leq C_1 \int_0^t \varphi(s)ds + (tC_2 + \varphi(0)), \qquad t \in (0, T).$$

Thus, by Gronwall's inequality,

$$\|u_\Gamma\|^2_{L^2(\Gamma)} \leq \left(tC_2 + \int_\Gamma u_0^2 dS \right) e^{C_1 t}, \qquad t \in (0, T).$$

Hence $u_\Gamma \in L^\infty(0, t^+; L^2(\Gamma))$, and therefore $u_\Gamma \in L^\infty((0, t^+) \times \Gamma)$ by (5.13). $\square$

Combing the Propositions 5.4 and 5.11 gives the following.

Theorem 5.12. *Let* (5.2) *hold, assume* $d \equiv d_*$ *and* $\delta \equiv \delta_* \geq 0$, *and that*

$$g(\xi) \sim \rho|\xi|^{q-1}\xi \qquad \text{as } |\xi| \to \infty,$$

for some $\rho \in \mathbb{R}$ *and* $q > 0$. *Then for all* $u_0 \in \mathcal{X}_\delta$ *the problem* (5.1) *has a unique global classical solution if and only if either* $\rho \leq 0$ *or* $q \leq 1$.

As for blow-up, we refine the sufficient conditions on g for global existence in case of the Laplace equation. We argue as in [4, Theorem 6.1], where the case $\delta \equiv 0$ was considered. The condition below complements the one of Proposition 5.6.

Proposition 5.13. *Let* (5.2) *hold, assume* $d \equiv d_*$ *and* $\delta \equiv \delta_* \geq 0$, *and that* $|g| \leq \gamma$, *where* $\gamma \in C(\mathbb{R}, (0, \infty))$ *is such that*

$$\int_0^\infty \frac{ds}{\gamma(s)} = \int_{-\infty}^0 \frac{ds}{\gamma(s)} = \infty.$$

Then for all $u_0 \in \mathcal{X}_\delta$ *the problem* (5.9) *has a unique global classical solution.*

Proof. Suppose that $t^+ < \infty$, and let $\xi(t)$, $\zeta(t) \in \overline{\Omega}$ be such that

$$m(t) := \min_{x \in \overline{\Omega}} u(t, x) = u(t, \xi(t)), \qquad M(t) := \max_{x \in \overline{\Omega}} u(t, x) = u(t, \zeta(t)).$$

It follows from [16, Theorem 3.5] that for each $t \in (0, t^+)$ we have $\xi(t), \zeta(t) \in \Gamma$. Thus $\partial_\nu u(t, \xi(t)) < 0$ and $\partial_\nu u(t, \zeta(t)) > 0$ by [16, Lemma 3.4]. By [4, Theorem 2.2], the function m is almost everywhere differentiable on $(0, t^+)$ with $\partial_t m = (\partial_t u)(t, \xi(t))$. Using that $\Delta_\Gamma u(t, \xi(t)) \geq 0$, which can be seen as in the proof of Lemma 5.3, we get

$$\partial_t m(t) = \delta \Delta_\Gamma u(t, \xi(t)) - \partial_\nu u(t, \xi(t)) + g(u(t, \xi(t))) \geq -\gamma(m(t))$$

for a.e. $t \in (0, t^+)$. In the same way we obtain $\partial_t M(t) \leq \gamma(M(t))$ for a.e. $t \in (0, t^+)$. Now the very same arguments as in the proof of [4, Theorem 3.1] provide a contradiction to the assumption $t^+ < \infty$. $\square$

5.4. **Global attractors.** Suppose that (5.2) and (5.12) hold true. Then by the above results, (5.1) generates a compact global solution semiflow

$$S_\delta(t;u_0) := u(t;u_0)$$

of smooth solutions in the phase space $\mathcal{X}_\delta$. Let $F' = f$ and $G' = g$. Then we may differentiate

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega d|\nabla u|^2\,dx + \frac{1}{2}\int_\Gamma \delta|\nabla_\Gamma u_\Gamma|^2\,dS - \int_\Omega (F(u) - \frac{\lambda}{2}u^2)\,dx - \int_\Gamma G(u_\Gamma)\,dS$$

with respect to time, to obtain

$$\partial_t \mathcal{E}(u) = -\|\partial_t u_\Gamma\|^2_{L^2(\Gamma)}. \tag{5.16}$$

Thus $\mathcal{E}$ is a strict Lyapunov function for (5.1), and the problem is of gradient structure. By [3, Corollary 1.1.7], for the existence of a global attractor it is left to show the boundedness of the set of equilibria E of (5.1). To formulate a sufficient condition for this, we note that by the global Lipschitz continuity of f there is a constant $\tilde{c}_f \in \mathbb{R}$ such that

$$f(\xi)\xi \le \tilde{c}_f(\xi^2 + 1), \qquad \xi \in \mathbb{R}. \tag{5.17}$$

For the boundedness of the equilibria the parameters of the problem should satisfy

$$\frac{\|\sqrt{d}\nabla\psi\|^2_{L^2(\Omega)} + \|\sqrt{\delta}\nabla_\Gamma\psi\|^2_{L^2(\Gamma)} - (\tilde{c}_f - \lambda)\|\psi\|^2_{L^2(\Omega)} - c_g\|\psi\|^2_{L^2(\Gamma)}}{\|\psi\|^2_{L^2(\Gamma)}} \ge \eta > 0, \tag{5.18}$$

for all $\Theta_\delta := \{\psi \in W^{1,2}(\Omega) : \psi|_\Gamma \in W^{1,2}(\Gamma) \text{ if } \delta \ge \delta_*\}$.

Lemma 5.14. *Assume* (5.2), (5.12), (5.17) *and* (5.18). *Then the set of equilibria* $E \subset C^\infty(\overline{\Omega})$ *of* (5.1) *is bounded in* $W^{2,p}(\Gamma)$ *for* $\delta \ge \delta_*$ *and it is bounded in* $W^{1,p}(\Gamma)$ *for* $\delta \equiv 0$.

Proof. *Step 1.* Note that indeed $E \subset C^\infty(\overline{\Omega})$ by Proposition 5.2. Thus an equilibrium u satisfies

$$\lambda u - \operatorname{div}(d\nabla u) = f(u) \quad \text{in } \Omega, \qquad -\operatorname{div}_\Gamma(\delta\nabla_\Gamma u) + d\partial_\nu u = g(u_\Gamma) \quad \text{on } \Gamma. \tag{5.19}$$

Multiplying by u, integrating by parts and using (5.12), (5.17) and (5.18), we get

$$\begin{aligned} C &\ge \|\sqrt{d}\nabla u\|^2_{L^2(\Omega)} + \|\sqrt{\delta}\nabla_\Gamma u_\Gamma\|^2_{L^2(\Gamma)} - (\tilde{c}_f - \lambda)\|u\|^2_{L^2(\Omega)} - c_g\|u_\Gamma\|^2_{L^2(\Gamma)} \\ &\ge \eta\|u_\Gamma\|^2_{L^2(\Gamma)}, \end{aligned}$$

with a constant C independent of u. Hence $\sup_{u\in E}\|u_\Gamma\|_{L^2(\Gamma)} < \infty$. Next we obtain from (5.15) that there are $C, \tau > 0$ such that

$$\int_\Gamma u_\Gamma^{2^k}\,dS \le 2^{k\tau}\int_\Gamma u_\Gamma^{2^{k-1}}\,dS + C$$

for all $u \in E$ and $k \in \mathbb{N}$. Hence, by an iteration argument,

$$\|u_\Gamma\|_{L^\infty(\Gamma)} \le C\big(1 + \|u_\Gamma\|_{L^2(\Gamma)}\big).$$

Therefore

$$\sup_{u\in E}\|u_\Gamma\|_{L^\infty(\Gamma)} < \infty. \tag{5.20}$$

Step 2. Suppose that $\delta \ge \delta_*$. Then for $u \in E$, (5.20) gives

$$\|u_\Gamma\|_{W^{2,p}(\Gamma)} \le C\big(\|u_\Gamma\|_{L^p(\Gamma)} + \|\Delta u_\Gamma\|_{L^p(\Gamma)}\big)$$

$$\leq C\big(1 + \|g(u_\Gamma)\|_{L^p(\Gamma)} + \|\partial_\nu u\|_{L^p(\Gamma)}\big) \leq C\big(1 + \|u\|_{H^{2-1/p,p}(\Omega)}\big).$$

Recall that $u = \mathcal{D}_\lambda(u_\Gamma)$, where $\mathcal{D}_\lambda : W^{2-2/p,p}(\Gamma) \to H^{2-1/p,p}(\Omega)$ is globally Lipschitzian by Lemma 4.4. Using the interpolation inequality, Young's inequality and (5.20), for arbitrary $\varepsilon > 0$ we get

$$\|u\|_{H^{2-1/p,p}(\Omega)} \leq C\big(1 + \|u_\Gamma\|_{W^{2-2/p,p}(\Gamma)}\big) \leq \varepsilon\|u_\Gamma\|_{W^{2,p}(\Gamma)} + C_\varepsilon,$$

where C_ε does not depend on $u \in E$. For small ε we can thus absorb $\varepsilon\|u_\Gamma\|_{W^{2,p}(\Gamma)}$ into the left-hand side of the previous inequality to obtain $\sup_{u\in E} \|u_\Gamma\|_{W^{2,p}(\Gamma)} < \infty$.

Step 3. Now let $\delta \equiv 0$. Then for $u \in E$ we have by Lemma 4.1 that

$$\begin{aligned}\|u_\Gamma\|_{W^{1,p}(\Gamma)} &\leq C\big(\|u_\Gamma\|_{L^p(\Gamma)} + \|\mathcal{N}_\lambda u_\Gamma\|_{L^p(\Gamma)}\big)\\ &\leq C\big(1 + \|\partial_\nu \mathcal{R}_\lambda(f(u), 0)\|_{L^p(\Gamma)}\big) \leq C\big(1 + \|u\|_{L^p(\Omega)}\big).\end{aligned}$$

Since $\|u\|_{L^p(\Omega)} \leq C\big(1 + \|u_\Gamma\|_{W^{1-1/p,p}(\Gamma)}\big)$ by Lemma 4.4, we may argue as above to obtain $\sup_{u\in E} \|u_\Gamma\|_{W^{1,p}(\Gamma)} < \infty$. ☐

Under the above assumptions it now follows from [3, Corollary 1.1.7] that the semiflow S_δ generated by (5.1) has a global attractor $\mathcal{A}_\delta$. To verify that $\mathcal{A}_\delta$ has finite Hausdorff dimension, we need the following.

Lemma 5.15. *Assume* (5.2) *and* (5.12). *Then for each $t > 0$ the time t map $S_\delta(t;\cdot)$ belongs to $C^\infty(\mathcal{X}_\delta)$, and the derivative $D_2 S_\delta(t;\cdot)$ is compact on $\mathcal{X}_\delta$.*

Proof. Recall that (5.1) may be rewritten into the form (5.3). The superposition operator induced by g belongs to $C^\infty(W^{s,p}(\Gamma), L^p(\Gamma))$ for all $s > \frac{n-1}{p}$. By Lemma 4.4, the same is true for $u_\Gamma \mapsto d\partial_\nu \mathcal{R}_\lambda(f(\mathcal{D}_\lambda(u_\Gamma)), 0)$. Therefore $S_\delta(t;\cdot)$ is smooth on $\mathcal{X}_\delta$ by e.g. [18, Corollary 3.4.5]. Since $S_\delta(t;\cdot)$ is a compact map by Theorem 4.8 and Proposition 4.13, also $D_2 S_\delta(t;\cdot)$ is compact. ☐

Since the global attractor $\mathcal{A}_\delta$ is by definition invariant under $S_\delta(1;\cdot)$, it is a consequence of [35, Chapter V, Theorem 3.2] that $\mathcal{A}_\delta$ has finite Hausdorff dimension.

We summarize the the results of this subsection as follows.

Theorem 5.16. *Assume* (5.2), (5.12), (5.17) *and* (5.18). *Then the solution semiflow S_δ on $\mathcal{X}_\delta$ for* (5.1) *has a global attractor $\mathcal{A}_\delta$, which is of finite Hausdorff dimension and coincides with the unstable set of equilibria.*

Remark 5.17. Another (more indirect) way to prove that $\mathcal{A}_\delta$ has finite fractal dimension is to establish the existence of a more refined object called exponential attractor $\mathcal{E}_\delta$, whose existence proof is based on the so-called smoothing property for the differences of any two solutions. This can be easily carried out in light of the assumptions for f, g, and the smoothness both in space and time for the solutions of (5.1) (see Proposition 5.2, and Lemma 5.24 below). It is also worth mentioning that the above result also holds for less regular functions in (5.2).

We conclude with a result that states a necessary and sufficient condition such that (5.18) is satisfied. To this purpose, consider the (self-adjoint) eigenvalue problem (see, e.g., [37, Theorem 2])

$$(\lambda - \widetilde{c}_f)\,\varphi - \operatorname{div}(d\nabla\varphi) = 0 \qquad \text{in } \Omega, \tag{5.21}$$

with a boundary condition that depends on the eigenvalue ξ explicitly,

$$-\operatorname{div}_\Gamma(\delta\nabla_\Gamma\varphi) + d\partial_\nu\varphi - c_g\varphi = \xi\varphi \qquad \text{on } \Gamma. \tag{5.22}$$

Proposition 5.18. *Let $\delta \geq 0$. Then inequality (5.18) is satisfied if and only if the first eigenvalue $\xi_1^\delta = \xi_1^\delta\left(\Omega, \widetilde{c}_f, c_g\right)$ of (5.21)-(5.22) is positive.*

Proof. We have that (see [37, Section 6])

$$\xi_1^\delta = \inf_{\psi \in \Theta_\delta, \psi \neq 0} \frac{\|\sqrt{d}\nabla\psi\|^2_{L^2(\Omega)} + \|\sqrt{\delta}\nabla_\Gamma\psi\|^2_{L^2(\Gamma)} - (\tilde{c}_f - \lambda)\|\psi\|^2_{L^2(\Omega)} - c_g\|\psi\|^2_{L^2(\Gamma)}}{\|\psi\|^2_{L^2(\Gamma)}},$$

from which the assertion immediately follows. □

Remark 5.19. It is easy to see that if $\lambda > \widetilde{c}_f$ and $c_g < C_P = C_P\left(\Omega, d, \lambda, \widetilde{c}_f\right)$, where $C_P > 0$ is the best constant in the following Poincaré-Sobolev type inequality

$$C_P\|\psi\|^2_{L^2(\Gamma)} \leq \|\sqrt{d}\nabla\psi\|^2_{L^2(\Omega)} + (\lambda - \widetilde{c}_f)\|\psi\|^2_{L^2(\Omega)},$$

then we always have $\xi_1^\delta > 0$.

5.5. **Convergence to single equilibria.** We shall finally be concerned with the asymptotic behavior of single trajectories. We first give sufficient conditions where a single homogeneous equilibrium is approached exponentially fast by every solution with respect to the $L^2(\Gamma)$-norm.

Proposition 5.20. *Assume* (5.2) *and that $f' \leq \tilde{c}_f$, $g' \leq c_g$ for $\tilde{c}_f, c_g \in \mathbb{R}$, such that* (5.18) *is valid. If* (5.1) *has a homogeneous equilibrium $u_* \in \mathbb{R}$, then for all $u_0 \in \mathcal{X}_\delta$ we have*

$$\|u(t;u_0) - u_*\|_{L^2(\Gamma)} \leq e^{-2\eta t}\|u_0 - u_*\|_{L^2(\Gamma)}, \qquad t > 0.$$

Proof. We first note that if (5.18) holds true, then at most one homogeneous equilibrium can exist, since it is necessary that either $\tilde{c}_f - \lambda < 0$ or $c_g < 0$.

It is straightforward to see that $g(\xi)\xi \leq (c_g + 1)\xi^2 + C$, such that every solution exists globally in time by Proposition 5.11. Let $w = u(\cdot, u_0) - u_*$. Testing the equations for w with w itself, we get

$$\begin{aligned}\frac{1}{2}\partial_t\|w_\Gamma\|^2_{L^2(\Gamma)} \leq &-\|\sqrt{d}\nabla w\|^2_{L^2(\Omega)} - \|\sqrt{\delta}\nabla_\Gamma w_\Gamma\|^2_{L^2(\Gamma)} \\ &+ (\tilde{c}_f - \lambda)\|w\|^2_{L^2(\Omega)} + c_g\|w_\Gamma\|^2_{L^2(\Gamma)}.\end{aligned}$$

Therefore $\partial_t\|w_\Gamma\|^2_{L^2(\Gamma)} \leq -2\eta\|w_\Gamma\|^2_{L^2(\Gamma)}$ by (5.18), and the result follows from Gronwall's inequality. □

We follow the approach of [34] to show the convergence of solutions to single equilibria also in nontrivial situations, under the assumption that f, g are real analytic. Thanks to Proposition 5.2 we can work with smooth solutions $u \in C^\infty\left((0, t^+) \times \overline{\Omega}\right)$.

In the situation of Theorem 5.16, the trajectory of any solution is bounded in $\mathcal{X}_\delta$, and thus relatively compact. Combining this with the gradient structure of (5.1), which is due to (5.16), we obtain the following properties of the limit sets

$$\omega(u_0) = \{u_* \in \mathcal{X}_\delta : \exists\, t_k \nearrow \infty \text{ such that } u(t_k; u_0) \to u_* \text{ as } k \to \infty\}$$

of trajectories (see, e.g., [35, Chapter I and Chapter VII]).

Lemma 5.21. *Assume* (5.2), (5.12), (5.17) *and* (5.18). *Then for any $u_0 \in \mathcal{X}_\delta$, the set $\omega(u_0)$ is a nonempty, compact and connected subset of $\mathcal{X}_\delta$. Furthermore, we have:*

(i) *$\omega(u_0)$ is fully invariant for the corresponding semiflow $S_\delta(t)$ on $\mathcal{X}_\delta$;*

(ii) *$\mathcal{E}$ is constant on $\omega(u_0)$;*

(iii) $\mathrm{dist}_{\mathcal{X}_\delta}(S_\delta(t)u_0, \omega(u_0)) \to 0$ *as* $t \to +\infty$;
(iv) $\omega(u_0)$ *consists of equilibria only.*

The key to prove that each solution converges to a single equilibrium in case when f and g are analytic is the following inequality of Łojasiewicz-Simon type.

Proposition 5.22. *Let* $d \equiv d_*$ *and* $\delta \in \{0,1\}$. *Assume that* f, g *are real analytic,* $|f'| \le c_f$, $\lambda > c_f$ *and that* (5.12) *and* (5.18) *are satisfied. Let* $u_* \in C^\infty(\overline{\Omega})$ *be an equilibrium of* (5.1). *Then there are constants* $\theta \in (0, 1/2)$ *and* $r > 0$, *depending on* u_*, *such that for any* $u \in C^2\left(\overline{\Omega}\right)$ *with* $\|u - u_*\|_{H^2(\Omega)} + \delta\|u_\Gamma - u_*\|_{H^2(\Gamma)} \le r$, *we have*

$$\begin{aligned}\|\lambda u - d\Delta u - f(u)\|_{L^2(\Omega)} &+ \|-\delta\Delta_\Gamma u_\Gamma + d\partial_\nu u - g(u_\Gamma)\|_{L^2(\Gamma)} \\ &\ge |\mathcal{E}(u) - \mathcal{E}(u_*)|^{1-\theta}.\end{aligned} \tag{5.23}$$

Proof. Our proof follows closely that of [34, Theorem 3.1] which only includes the case $\delta = 1$ (cf. also [42] for $g \equiv 0$ and $\delta = 0$). We shall briefly mention the details below in the case when $\delta = 0$ and g is nontrivial. To this end, let us first set

$$V_k := \left\{(u, u_\Gamma) \in W^{k,2}(\Omega) \times W^{k-1/2,2}(\Gamma) : u_\Gamma = u|_\Gamma\right\},$$

for $k \ge 1$ (by convention, we also let $V_0 := L^2(\Omega) \times L^2(\Gamma)$). Here and below, for the sake of simplicity of notation we will identify any function u that belongs to $W^{k,2}(\Omega)$ with $(u, u_\Gamma) \in V_k$ such that $u_\Gamma \in W^{k-1/2,2}(\Gamma)$. Next, consider the so-called Wentzell Laplacian, given by

$$A_W := \begin{pmatrix} \lambda I - d\Delta & 0 \\ d\partial_\nu & 0 \end{pmatrix},$$

with domain $D(A_W) = \left\{u \in V_1 : -\Delta u \in L^2(\Omega),\ \partial_\nu u \in L^2(\Gamma)\right\}$, which we endow with its natural graph norm $\|A_W \cdot\|_{V_0}$. In particular, $D(A_W) = V_2$ provided that the boundary Γ is sufficiently regular (see [14], [15]). It is also well-known that $(A_W, D(A_W))$ is self-adjoint and positive on V_0. By [13], we also infer that there exists a complete orthonormal family $\{\phi_j\} \subset V_0$, with $\phi_j \in D(A_W)$, as well as a sequence of eigenvalues $0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_j \to \infty$, as $j \to \infty$, such that $A_W\phi_j = \lambda_j\phi_j$, $j \in \mathbb{N}_+$. Moreover, by standard elliptic theory and bootstrap arguments we have $\phi_j \in C^\infty$, for every $j \in \mathbb{N}_+$ provided that Γ is sufficiently regular (see [13, Appendix]). Let now P_m be the orthogonal projector from V_0 onto $K_m := span\{\phi_1, ..., \phi_m\}$. Following a similar strategy to [34, (3.8)-(3.11)], it is easy to show that

$$(A_W u + \lambda_m P_m u, u)_{V_1^*, V_1} \ge C_{\lambda,d}\|u\|_{V_1}^2 + \frac{1}{4}\lambda_m\|u\|_{V_0}^2 \tag{5.24}$$

holds for any $u \in V_1$ (for some positive constant $C_{\lambda,d} > 0$).

Let ψ be a critical point of $\mathcal{E}(u)$. For any $\psi \in C^2\left(\overline{\Omega}\right)$, we consider the following linearized operator $v \in V_2 \longmapsto L(v)$, analogous to [34, (3.12)], given by

$$L(v) := \begin{pmatrix} \lambda I - d\Delta & 0 \\ d\partial_\nu & 0 \end{pmatrix} + \begin{pmatrix} -f'(v+\psi) & 0 \\ 0 & -g'(v+\psi) \end{pmatrix},$$

with domain $\mathcal{D} := D(A_W) = V_2$. We note that one can associate with $L(0)$ the following bilinear form $b(u_1, u_2)$ on $V_1 \times V_1$, as follows:

$$b(u_1, u_2) = \int_\Omega \left(\lambda u_1 u_2 + d\nabla u_1 \cdot \nabla u_2 - f'(\psi)u_1u_2\right)dx + \int_\Gamma \left(-g'(\psi)u_1u_2\right)dS,$$

for any $u_1, u_2 \in V_1$. As in [15], it can be easily shown that $(L(v), \mathcal{D})$ is self-adjoint on V_0. Moreover, by (5.24) it is readily seen that the operator $L(0) + \lambda_m P_m$ is coercive with respect to the (equivalent) inner product of $H^1(\Omega)$, provided that

$$\lambda_m > 4 \max \left\{ \left\| f^{'}(\psi) \right\|_{L^\infty(\Omega)}, \left\| g^{'}(\psi) \right\|_{L^\infty(\Gamma)} \right\}. \tag{5.25}$$

Recalling that ψ is sufficiently smooth, we note that condition (5.25) can always be achieved by choosing a sufficiently large m.

Next, consider the following operators:

$$\Pi := \lambda_m P_m : V_0 \to V_0,\ \mathcal{L}(v) : \mathcal{D} \to V_0,\ \mathcal{L}(v) h = \Pi h + L(v) h,$$

for any $v \in \mathcal{D}$. Clearly, $\mathcal{L}(0) : \mathcal{D} \to V_0$ is bijective on account of (5.25). The following lemma is concerned with some regularity properties for the (unique) solution of $\mathcal{L}(0) h = w$, for some given $w = (w_1, w_2) \in V_0$.

Lemma 5.23. *Let $w = (w_1, w_2) \in W^{k-1,2}(\Omega) \times W^{k-1/2,2}(\Gamma)$, for any $k \in \mathbb{N}$, $k \geq 1$. Then the following estimate holds:*

$$\|h\|_{V_{k+1}} \leq C \left(\|w_1\|_{W^{k-1,2}(\Omega)} + \|w_2\|_{W^{k-1/2,2}(\Gamma)} \right). \tag{5.26}$$

Moreover, it holds

$$\|h\|_{C^{0,\gamma}(\overline{\Omega})} \leq C \left(\|w_1\|_{L^p(\Omega)} + \|w_2\|_{L^q(\Gamma)} \right), \tag{5.27}$$

as long as $w_1 \in L^p(\Omega)$ with $p > n$ and $w_2 \in L^q(\Gamma)$ with $q > n-1$. The constant $C > 0$ is independent of k.

Indeed, writing the equation $\mathcal{L}(0) h = w$ in the form

$$\begin{cases} \lambda h - d\Delta h = q_1 := f^{'}(\psi) h - \Pi h + w_1, & \text{a.e. in } \Omega, \\ d\partial_\nu h = q_2 := g^{'}(\psi) h - (\Pi h)_{|\Gamma} + w_2, & \text{a.e. on } \Gamma, \end{cases} \tag{5.28}$$

we have $\|h\|_{V_1 \cap \mathcal{D}} \leq C \|w\|_{V_0}$ since $\mathcal{L}(0) : \mathcal{D} \to V_0$ is a bijection. Recalling the standard trace-regularity theory and the fact that $\psi \in C^\infty$, and applying the $W^{l,2}$ regularity theorem (see, e.g., [23, II, Theorem IV.5.1]) to (5.28) with $l = 2, 3, ...$, we have

$$\|h\|_{V_l} \leq C \|h\|_{W^{l,2}(\Omega)} \leq C \left(\|q_1\|_{W^{l-2,2}(\Omega)} + \|h\|_{W^{l-1,2}(\Omega)} + \|q_2\|_{W^{l-3/2,2}(\Gamma)} \right).$$

From this (5.26) immediately follows by exploiting a standard iteration argument for $l \geq 2$. The proof of (5.27) is contained in [40] (see, also, [41] for the proof of the same bound in $L^\infty(\Omega) \times L^\infty(\Gamma)$-norm).

Exploiting now the results of the preceding lemma, the proof of the (extended) Łojasiewicz-Simon inequality (5.23) can be reproduced from that of [34, Theorem 3.1 and Lemma 3.2] with no essential modifications. □

To apply the proposition we need that the solutions converge to the limit set in a norm that is stronger than that of $\mathcal{X}_\delta$. This will be a consequence of the next lemma. Let $B_{\mathcal{X}_\delta}(R)$ be the ball in $\mathcal{X}_\delta$ centered at the origin with radius $R > 0$.

Lemma 5.24. *Assume (5.2), (5.12), (5.17) and (5.18). Then for $k \in \mathbb{N}_0$ and $R > 0$ there is a constant $C = C(k, R) > 0$ such that*

$$\sup_{u_0 \in B_{\mathcal{X}_\delta}(R)} \sup_{t \geq 1} \|S_\delta(t; u_0)\|_{C^k(\overline{\Omega})} \leq C.$$

Proof. By Theorem 5.16, the semiflow has global attractor in $\mathcal{X}_\delta$. Therefore, using Lemma 4.4,

$$\|u\|_{L^\infty(\mathbb{R}_+;H^{2-1/p,p}(\Omega))} \le C(R) \qquad \text{if } \delta \ge \delta_*,$$
$$\|u\|_{L^\infty(\mathbb{R}_+;W^{1,p}(\Omega))} \le C(R) \qquad \text{if } \delta \equiv 0,$$

for all $u = u(\cdot;u_0)$ with $u_0 \in B_{\mathcal{X}_\delta}(R)$. Let us consider the case $\delta \equiv 0$. We use the representation

$$u_\Gamma(t) = e^{-t\mathcal{N}_\lambda}u_0 + e^{-\cdot\mathcal{N}_\lambda} * \big(g(u_\Gamma) - d\partial_\nu \mathcal{R}_\lambda(f(u),0)\big)(t), \qquad t>0, \tag{5.29}$$

and assume, inductively, that

$$\|u\|_{L^\infty(1,\infty;W^{k,p}(\Omega))} \le C(k,R)$$

for some $k\in\mathbb{N}$. Then $\|g(u_\Gamma)\|_{L^\infty(1,\infty;W^{k-1/p,p}(\Gamma))} \le C(k,R)$, and further, using [1, Theorem 13.1],

$$\|d\partial_\nu\mathcal{R}_\lambda(f(u),0)\|_{L^\infty(1,\infty;W^{k-1/p,p}(\Gamma))} \le C(k)\|f(u)\|_{L^\infty(1,\infty;W^{k-1,p}(\Omega))} \le C(k,R).$$

Considering (5.29) as an identity on $W^{k-1/p,p}(\Gamma)$, it follows from [25, Proposition 4.4.1] that $\|u_\Gamma\|_{L^\infty(1,\infty;W^{k+1-1/p,p}(\Gamma))} \le C(k+1,R)$. Moreover, by [1, Theorem 13.1], we have

$$\begin{aligned}\|u\|_{L^\infty(1,\infty;W^{k+1,p}(\Omega))} &\le C\big(\|f(u)\|_{L^\infty(1,\infty;W^{k-1,p}(\Omega))} + \|u_\Gamma\|_{L^\infty(1,\infty;W^{k+1-1/p,p}(\Gamma))}\big)\\ &\le C(k+1,R).\end{aligned}$$

The asserted estimate in case $\delta\equiv 0$ now follows from Sobolev's embeddings. The arguments in the case $\delta \ge \delta_*$ are similar. □

We can now prove the main result of this subsection.

Theorem 5.25. *Let $p\in(n,\infty)$, $d>0$ and $\delta\in\{0,1\}$. Assume that f,g are real analytic, $|f'|\le c_f$, $\lambda > c_f$ and that g satisfies* (5.12) *and* (5.18). *Then for any given initial datum $u_0\in\mathcal{X}_\delta$ the corresponding solution $u(t;u_0) = S_\delta(t;u_0)$ of (5.1) exists globally in time and converges to a single equilibrium u_* in the topology of $\mathcal{X}_\delta$. More precisely,*

$$\lim_{t\to+\infty}\big(\|u(t;u_0)-u_*\|_{\mathcal{X}_\delta} + \|\partial_t u(t;u_0)\|_{L^2(\Gamma)}\big) = 0. \tag{5.30}$$

Proof. *Step 1.* From Theorem 5.16 and Lemma 5.21 we know that the solution $u=u(\cdot;u_0)$ is smooth in space and time, exists globally and that the corresponding trajectory converges to the set of equilibria in $\mathcal{X}_\delta$. By Lemma 5.24, the trajectory is also bounded in, say, $W^{3,p}(\Gamma)$. Since $\omega(u_0)\subset C^\infty(\overline{\Omega})$, we can apply the interpolation inequality (2.1) with suitable $\theta\in(0,1)$ to obtain that

$$\mathrm{dist}_{\mathcal{V}_\delta}(u(t;u_0),\omega(u_0)) \to 0 \qquad \text{as } t\to+\infty, \tag{5.31}$$

where we have set $\mathcal{V}_0 := H^2(\Omega)$ if $\delta=0$, and $\mathcal{V}_1 := H^2(\Omega)\oplus H^2(\Gamma)$ if $\delta=1$, respectively.

Step 2. The function $t\mapsto\mathcal{E}(u(t))$ is decreasing and bounded from below. Thus

$$\mathcal{E}_\infty := \lim_{t\to\infty}\mathcal{E}(u(t))$$

exists. If there is $t^\sharp$ with $\mathcal{E}(u(t^\sharp)) = \mathcal{E}_\infty$, then u is an equilibrium and there is nothing to prove. Hence we may suppose that for all $t\ge t_0>0$, we have $\mathcal{E}(u(t)) > \mathcal{E}_\infty$. We first observe that, by Lemma 5.22, the functional $\mathcal{E}$ satisfies the Łojasiewicz-Simon inequality (5.23) near every $u_*\in\omega(u_0)$. Since $\omega(u_0)$ is compact in $\mathcal{X}_\delta$, we can cover it by the union of finitely many balls B_j with centers u_*^j and radii

r_j, where each radius is such that (5.23) holds in B_j. It follows from Proposition 5.22 that there exist uniform constants $\xi \in (0,1/2)$, $C_L > 0$ and a neighborhood U of $\omega(u_0)$ in $\mathcal{X}_\delta$ such that (5.23) holds in U. Thus, recalling (5.31), we can find a time $t_0 \geq 1$ such that $u(t;u_0)$ belongs to U for all $t \geq t_0$. On account of (5.16) and (5.23) we obtain

$$\begin{aligned} &-\frac{d}{dt}(\mathcal{E}(u(t)) - \mathcal{E}_\infty)^\xi \\ &= -\xi \partial_t \mathcal{E}(u(t))(\mathcal{E}(u(t)) - \mathcal{E}_\infty)^{\xi-1} \\ &\geq \xi C_L \frac{\|\partial_t u_\Gamma\|^2_{L^2(\Gamma)}}{\|\lambda u - d\Delta u - f(u)\|_{L^2(\Omega)} + \|-\delta\Delta_\Gamma u_\Gamma + d\partial_\nu u - g(u_\Gamma)\|_{L^2(\Gamma)}}. \end{aligned}$$

Recalling (5.1), we get

$$-\frac{d}{dt}\left(\mathcal{E}\left(u\left(t\right)\right) - \mathcal{E}_\infty\right)^\xi \geq C\left\|\partial_t u_\Gamma(t)\right\|_{L^2(\Gamma)}. \tag{5.32}$$

Integrating over (t_0,∞) and using that $\mathcal{E}(u(t)) \to \mathcal{E}_\infty$ as $t \to \infty$, we infer that

$$\partial_t u_\Gamma \in L^1\left(t_0,\infty; L^2\left(\Gamma\right)\right).$$

Step 3. Since $\partial_t u_\Gamma$ is uniformly continuous with values in $L^2(\Gamma)$, it follows that $\|\partial_t u_\Gamma(t)\|_{L^2(\Gamma)} \to 0$ as $t \to \infty$. Moreover, since by Lemma 5.21 (iii) there are $t_k \nearrow \infty$ and $u_* \in \omega\left(u_0\right)$ such that $u(t_k) \to u_*$ in $\mathcal{X}_\delta$ as $k \to \infty$, the integrability of $\partial_t u_\Gamma$ implies that $u_\Gamma(t) \to u_*$ in $L^2(\Gamma)$ as $t \to \infty$, and then in $\mathcal{X}_\delta$ as well. Hence $\omega(u_0) = \{u_*\}$, and (5.30) follows. □

Remark 5.26. One can also exploit (5.23) and (5.32) to deduce a convergence rate estimate in (5.30) of the form

$$\|u(t;u_0) - u_*\|_{L^2(\Gamma)} \leq C(1+t)^{-1/(1-2\xi)}, \qquad t > 0,$$

for some constants $C > 0$, $\xi \in \left(0, \frac{1}{2}\right)$ depending on u_*. Taking advantage of the above (lower-order) convergence estimate and the results of the previous subsections, one can also prove the corresponding estimate in higher-order norms $W^{k,2}(\Omega)$, arguing, for instance, as in [34, 42].

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